

CTP TAMU-xx/00
 May 2000
hep-th/0005172

Kaluza-Klein Consistency, Killing Vectors and Kähler Spaces

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ABSTRACT

We make a detailed investigation of all spaces $Q_{n_1 \dots n_N}^{q_1 \dots q_N}$ of the form of $U(1)$ bundles over arbitrary products $\prod_i CP^{n_i}$ of complex projective spaces, with arbitrary winding numbers q_i over each factor in the base. Special cases, including Q_{11}^{11} (sometimes known as T^{11}), Q_{111}^{111} and Q_{21}^{32} , are relevant for compactifications of type IIB and $D = 11$ supergravity. Remarkable “conspiracies” allow consistent Kaluza-Klein S^5 , S^4 and S^7 sphere reductions of these theories that retain all the Yang-Mills fields of the isometry group in a massless truncation. We prove that such conspiracies do not occur for the reductions on the $Q_{n_1 \dots n_N}^{q_1 \dots q_N}$ spaces, and that it is *inconsistent* to make a massless truncation in which the non-abelian $SU(n_i + 1)$ factors in their isometry groups are retained. In the course of proving this we derive many properties of the spaces $Q_{n_1 \dots n_N}^{q_1 \dots q_N}$ of more general utility. In particular, we show that they always admit Einstein metrics, and that the spaces where $q_i = (n_i + 1)/\ell$ all admit two Killing spinors. We also obtain an iterative construction for real metrics on CP^n , and construct the Killing vectors on $Q_{n_1 \dots n_N}^{q_1 \dots q_N}$ in terms of scalar eigenfunctions on CP^{n_i} . We derive bounds that allow us to prove that certain Killing-vector identities on spheres, necessary for consistent Kaluza-Klein reductions, are never satisfied on $Q_{n_1 \dots n_N}^{q_1 \dots q_N}$.

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1 Introduction

In its original form Kaluza-Klein reduction was used for the purpose of deriving a four-dimensional theory comprising gravity, a $U(1)$ gauge field and a dilatonic scalar, starting from pure gravity in five dimensions. The extra dimension is taken to be a circle, and the five-dimensional metric is then assumed to be independent of the coordinate y on the circle. Such a truncation is consistent, and gives rise to an Einstein-Maxwell theory in $D = 4$, coupled to the dilatonic scalar field. The consistency of the truncation is assured because the reduction ansatz retains all the four-dimensional fields that are independent of y , while setting all fields that would be associated with y -dependent harmonics on S^1 to zero. In a similar vein, Kaluza-Klein reductions involving higher-dimensional theories compactified on tori can also be considered, and again consistent truncations where all fields are taken to be independent of the torus coordinates can be performed.

The situation is much less clear-cut in the case where one performs a reduction on a curved internal manifold, such as a sphere. The new complication in such a case is that the harmonics on the internal space associated with the massless fields in the lower dimension typically now depend on the coordinates of the internal space. This causes no difficulty in a linearised analysis of small fluctuations around a ground-state solution (see, for example, [1], and references therein), but as soon as one wants to consider the full non-linear structure of the theory it raises the possibility of inconsistencies in a truncation to the massless sector. In fact this is more than a possibility; in general, there will definitely be inconsistencies. This makes it all the more remarkable that there exist certain exceptional cases in which a fully non-linear sphere reduction and truncation *is* completely and rigorously consistent. Many of the known cases involve special reductions of supergravity theories, notably involving S^7 [2] or S^4 [3, 4] reductions of $D = 11$, the S^5 reduction of type IIB,¹ and a local S^4 reduction of the massive type IIA theory [9]. Other exceptional examples of consistent sphere reductions in which all the Yang-Mills gauge fields can be retained have also been found recently, for cases that do not necessarily have any connection with supersymmetry. These comprise the reduction of the low-energy limit of the bosonic string, in an arbitrary dimension D , on the 3-sphere or the $(D - 3)$ -sphere, and the reduction of certain theories of gravity plus a dilaton and a 2-form field strength in D dimensions on a 2-sphere [10]. In all these cases,

¹The consistency of the S^5 reduction to five-dimensional maximal gauged supergravity remains conjectural at this time, but strong supporting evidence has been obtained, including various explicit consistent reductions to subsets of the maximal supergravity [5, 6, 7], and an explicit expression for the complete metric reduction ansatz [8].

there is no known group-theoretic proof for why the reduction should be consistent.²

Two approaches to proving the consistency of these supergravity sphere reductions have been pursued in the literature. For the S^7 [2] and S^4 [3, 4] reductions from $D = 11$, the truncations to the maximally supersymmetric gauged $SO(8)$ and $SO(5)$ supergravities in $D = 4$ and $D = 7$ have been argued to be consistent by demonstrating that consistent supersymmetry transformation rules in the lower dimension can be extracted from the original ones in $D = 11$. A complete and rigorous proof of consistency along these lines would in principle require the analysis of the supersymmetry transformation rules to all orders, including quartic fermion terms, and the difficulties in doing this are considerable. However, it seems reasonable to conclude that the already highly non-trivial success at the quadratic level would persist to all orders. The approach has been used for the S^7 [2] and S^4 [3, 4] reductions, and in the latter case has allowed an explicit construction of the exact bosonic reduction ansatz. No analogous complete results have been obtained for the S^5 reduction of type IIB supergravity, but it seems highly likely to be consistent also.

The alternative approach to proving the consistency of a Kaluza-Klein reduction is a more direct one, in which one explicitly constructs a reduction ansatz which, when substituted into the full set of higher-dimensional equations of motion, gives a consistent embedding provided that the lower-dimensional equations of motion are satisfied. This approach has been used to provide a complete proof of the consistency in several sphere reductions, where further truncations to subsets of the fields of the maximal massless supermultiplet are made. Cases that have been fully proven by this means include $N = 2$ gauged $SU(2)$ supergravity in $D = 7$ by an S^4 reduction from $D = 11$ [11]; the $N = 4$ gauged $SU(2) \times U(1)$ supergravity in $D = 5$ by an S^5 reduction from type IIB in $D = 10$ [6]; the $N = 4$ gauged $SO(4)$ supergravity by S^7 reduction from $D = 11$ [12], and the $N = 2$ gauged $SU(2)$ supergravity in $D = 6$ by a local S^4 reduction from the massive type IIA theory in $D = 10$ [9]. (In this last example $N = 2$ is in fact the largest supersymmetry for gauged supergravity in $D = 6$, even though ungauged $N = 4$ supergravity exists.) In addition, the consistency of the truncations of the S^4 , S^5 and S^7 reductions to include gravity and all the diagonal scalars of the $SL(5, R)/SO(5)$, $SL(6, R)/SO(6)$ and $SL(8, R)/SO(8)$ submanifolds of the full scalar cosets of the maximal supergravities [13] have been fully demonstrated [14].

It is significant that all these examples involve reductions on spheres. At the linearised

²A group-theoretic argument has been used in [7, 10] in order to prove that an n -sphere reduction of a theory of gravity plus dilaton plus n -form field strength that retained all the $SO(n+1)$ Yang-Mills fields in a massless truncation could not be consistent except in the exceptional cases listed above.

level there is no reason why one should not consider also reductions on internal spaces of other topologies. Examples that have been considered in the past include the Einstein spaces constructed as $U(1)$ bundles over $CP^2 \times S^2$ and $S^2 \times S^2 \times S^2$, as compactifications of eleven-dimensional supergravity, and $U(1)$ bundles over $S^2 \times S^2$, as compactifications of type IIB supergravity. The first two examples were first discussed in detail in [15, 16], where they were constructed as the coset spaces $M^{pqr} = SU(3) \times SU(2) \times U(1)/(SU(2) \times U(1) \times U(1))$ and $Q^{pqr} = SU(2)^3/(U(1) \times U(1))$ respectively, and the linearised massless spectra were obtained. They were subsequently reconstructed from the viewpoint of $U(1)$ bundles over $CP^2 \times S^2$ and $S^2 \times S^2 \times S^2$ respectively, in [19, 20], where a stability analysis was also given. The complete massive spectrum was obtained in a linearised analysis in [17, 18, 21, 22]. The five-dimensional example, the $U(1)$ bundle over $S^2 \times S^2$, was discussed from the AdS/CFT viewpoint in [23], and in a field-theoretic context in [24]. The full Klauza-Klein spectrum was obtained in [25], and its matching with the conformal operators of the dual CFT was obtained.

Amongst the lower-dimensional massless fields that would result from reductions such as these will be Yang-Mills gauge bosons with gauge group given by the isometry group of the internal space. One may wonder whether a consistent truncation that includes the Yang-Mills gauge fields is possible in these more general reductions too, or whether it is a special feature of the spherical spaces that ensures the consistency.

Some results on certain of these more general reductions were obtained in previous studies. In this paper, we shall address the question in a slightly broader context. The conclusions will be similar to those reached in the previous cases, namely that in general the reductions on non-spherical internal spaces do not allow consistent truncations to the massless sector, even in those exceptional and remarkable theories where consistent sphere reductions are possible. In fact it is much easier to demonstrate the inconsistency of an inconsistent truncation than to prove the consistency of a consistent one. As was discussed in [26], when there are inconsistencies they tend to show up in relatively easily-studied sectors of the theory, at the level of cubic interaction terms in the Lagrangian. In this paper we shall be considering a specific type of cubic interaction, namely terms that are bilinear in the lower-dimensional gauge fields, and that couple to lower-dimensional linearised spin-2 fields. This sector provides a necessary condition for consistency of a truncation; in general it turns out that the bilinears in gauge fields can act as sources for *massive* as well as massless spin-2 excitations. If this happens, then setting the massive spin-2 fields to zero is inconsistent with the higher-dimensional equations of motion, and so the reduction is

established to be an inconsistent one. We derive this condition in section 2.

As we shall discuss, the absence or presence of these kinds of trilinear couplings is governed by whether or not the Killing vectors on the internal space satisfy a certain quadratic identity. We shall show that although the full set of $SO(n+1)$ Killing vectors on the sphere S^n do indeed satisfy the identity, implying no inconsistency in this sector, the full sets of Killing vectors in the case of other internal manifolds do not. In particular, we shall show by this means that for the 5-dimensional space $Q(1,1)$ (sometimes called T^{11}), which can be described as a $U(1)$ bundle over $S^2 \times S^2$, only the Killing vector of the $U(1)$ factor in its $U(1) \times SU(2) \times SU(2)$ isometry group satisfies the consistency condition. Thus in a reduction of type IIB supergravity on the $Q(1,1)$ space, only the $U(1)$ gauge field of the $N=2$ supergravity multiplet can be consistently retained in a massless truncation, whilst the $SU(2) \times SU(2)$ gauge fields of the matter multiplets must be set to zero.

In order to demonstrate that the $SU(2) \times SU(2)$ Killing vectors of the $Q(1,1)$ space fail to satisfy the consistency criterion, it is helpful to obtain an explicit construction for them. Motivated by this, we have undertaken a rather more general investigation of the construction of Killing vectors in spaces of this kind. The base space $S^2 \times S^2$ in the construction of $Q(1,1)$ as a $U(1)$ bundle is Kähler, and in fact in this specific case it itself is an Einstein space. More generally, one can consider the $U(1)$ bundle spaces over any Einstein-Kähler base space, or over a product of Einstein-Kähler spaces. Other relevant examples of this kind are the 7-dimensional $M(3,2)$ and $Q(1,1,1)$ spaces that have been used in compactifications of $D=11$ supergravity [15, 16]. These arise, respectively, as $U(1)$ bundles over $CP^2 \times S^2$ and over $S^2 \times S^2 \times S^2$ [19, 20]. In all the cases, the curvature of the $U(1)$ connection is proportional to the sum of the Kähler forms on the factors in the base space.

Intuitively, one expects that if the base space has an isometry group G , and the curvature of the $U(1)$ bundle is invariant under G , then the isometry group of the bundle space should be at least $U(1) \times G$. In section 3 we show how to make this precise, and we obtain explicit formulae that allow one to “lift” the Killing vectors of the base space to Killing vectors in the bundle space. The situation is especially nice if the base space is Einstein-Kähler, or else a product of Einstein-Kähler spaces, and we show how one can then express the Killing vectors of the base, and hence of the bundle space, in terms of certain scalar harmonics on the Einstein-Kähler factors in the base space.

In section 4 we specialise the discussion to the case where the Einstein-Kähler manifolds M_i in the product base space $M = M_1 \times M_2 \times M_N$ are taken to be $M_i = CP^{n_i}$. We denote

the corresponding bundle spaces by $Q_{n_1 n_2 \dots n_N}^{q_1 q_2 \dots q_N}$, where q_i is the winding number of the $U(1)$ fibre over the factor $M_i = \mathbb{C}P^{n_i}$ in the product base manifold. The three examples $Q(1, 1)$, $M(3, 2)$ and $Q(1, 1, 1)$ described above are special cases within this general class, namely Q_{11}^{11} , Q_{21}^{32} and Q_{111}^{111} respectively. We give an explicit construction of the $SU(n+1)$ Killing vectors of $\mathbb{C}P^n$ in terms of certain scalar harmonics. Using this construction and the results from section 3, we are able to lift all the Killing vectors of the product base manifold for $Q_{n_1 n_2 \dots n_N}^{q_1 q_2 \dots q_N}$ into the total bundle space, thereby exhibiting its $U(1) \times \prod_i SU(n_i + 1)$ isometry group.

We prove also that all the $U(1)$ bundle spaces $Q_{n_1 n_2 \dots n_N}^{q_1 q_2 \dots q_N}$ admit Einstein metrics of positive Ricci curvature, for all possible choices of the winding numbers q_i , provided only that they do not all vanish. In addition we show that when all the q_i are given by $q_i = n_i + 1$, the Einstein metric admits 2 Killing spinors.

In section 5 we make use of some of the general results from sections 3 and 4, to show explicitly that the Killing vectors in the $SU(n_i + 1)$ factors in spaces such Q_{11}^{11} , Q_{21}^{32} and Q_{111}^{111} do not satisfy the consistency criterion for the Kaluza-Klein reductions. These results support the suggestion, made in [27], that only the massless fields in the *supergravity* multiplet, as opposed to any massless *matter* multiplets, can be consistently retained in a Kaluza-Klein reduction using a curved internal space. Thus the reason why spheres work so well in Kaluza-Klein supergravity reductions is because they maximise the number of Killing spinors, and thus their supergravity multiplets are larger than those for any other choice of compactifying space. More generally, in section 5, we analyse the analogue of the consistency condition for bundle spaces of arbitrary dimension, and we show that always the Killing vectors associated with the isometries of the base manifold, when it is a product of two or more complex projective spaces, do not satisfy the consistency condition. Two appendices contain some further general results, including an iterative construction of real metrics on $\mathbb{C}P^n$, and a detailed analysis of certain bounds on integrals involving the scalar eigenfunctions on $\mathbb{C}P^n$, which are needed for the results in section 5.

2 Consistency conditions on Killing vectors in Kaluza-Klein reductions

In this section, we shall focus principally on the Kaluza-Klein reduction of type IIB supergravity on a 5-dimensional internal space \mathcal{M}_5 . Analogous results have previously been obtained for reductions of $D = 11$ supergravity [28], and we shall mention these briefly at

the end of the section.

Since our goal will be to derive a *necessary* condition for the consistency of the reduction, with a view to showing that the condition is not in fact satisfied except in very special circumstances, it will be sufficient to carry out an analysis that is based on a linearised approximation. Thus we shall consider a situation where \mathcal{M}_5 is an Einstein space of positive Ricci curvature, and we shall consider small fluctuations around the $\text{AdS}_5 \times \mathcal{M}_5$ Freund-Rubin background. In particular, we shall consider the Yang-Mills gauge bosons associated with the isometry group G of the internal space \mathcal{M}_5 .

Although we shall consider only the linearised ansatz for the gauge bosons this will actually enable us to consider the effects of non-linear terms in theory, and in particular to show that bilinears in the gauge fields will in general act as sources for massive spin-2 fields. The reason why we can use a linearised ansatz for this purpose is that gauge invariance ensures that there can be no additional contributions from a full non-linear reduction ansatz that could “help out” and resolve the consistency problems that we shall be able to reveal. Thus, since our goal is only to prove inconsistency, not consistency, the analysis presented here will be sufficient.³

The fields of the type IIB theory that are relevant for this discussion are the metric tensor \hat{G}_{MN} and the self-dual 5-form field strength \hat{H}_5 , which we may write as $\hat{H}_5 = \hat{G}_5 + \hat{*}\hat{G}_5$. The type IIB equations of motion for these fields are then

$$\begin{aligned}\hat{R}_{MN} &= \frac{1}{96} \hat{H}_{MPQRS} \hat{H}_N{}^{PQRS}, \\ d\hat{H}_5 &= d * \hat{H}_5 = 0.\end{aligned}\tag{2.1}$$

The Freund-Rubin $\text{AdS}_5 \times \mathcal{M}_5$ ground-state solution is then obtained by setting $\hat{G}_5 = 4m\epsilon_5$, where ϵ_5 is the spacetime volume form and m is a constant. The equations of motion are then satisfied if the Ricci tensors in the five-dimensional spacetime and the internal space \mathcal{M}_5 satisfy

$$R_{\mu\nu} = -4m^2 g_{\mu\nu}, \quad \text{and} \quad R_{mn} = 4m^2 g_{mn}\tag{2.2}$$

³Note that we shall ignore the contributions of other five-dimensional fields, including the scalar fields, in this discussion. Truncating out these fields, while keeping the Yang-Mills gauge fields, is itself an inconsistent procedure, since the Yang-Mills fields would in principle act as sources for them. The point is, though, that these are quite distinct and separate inconsistencies, which would show up in different sectors of the theory. By focusing, as we shall, on the five-dimensional spacetime components of the ten-dimensional Einstein equation we shall be able to isolate a particular inconsistency that is independent of the neglect of the other fields. In other words, including the other fields in the ansatz would not help to resolve the inconsistency that we shall exhibit.

respectively. Thus we may take the spacetime metric to be AdS_5 with cosmological constant $-4m^2$, and \mathcal{M}_5 to be any 5-dimensional Einstein space with positive cosmological constant⁴ $4m^2$.

We may now consider the contributions of the five-dimensional Yang-Mills gauged bosons in the ansätze for the ten-dimensional metric and 5-form field strength, at the leading-order linearised level. For the metric, this will be

$$d\hat{s}^2 = e^\alpha e^\beta \eta_{\alpha\beta} + (e^a - K^{Ia} A^I)(e^b - K^{Jb} A^J) \delta_{ab}, \quad (2.3)$$

where $e^\alpha = e^\alpha(x)$ is the vielbein in the $d = 5$ spacetime, $e^a = e^a(y)$ is the vielbein in the internal space, $K^{Ia} = K^{Ia}(y) = K^{Im}(y) e_m{}^a(y)$ are the orthonormal components of the Killing vectors which generate the isometry group G of the internal space \mathcal{M}_5 , and $A^I = A^I(x) = e^\alpha(x) A_\alpha^I(x)$ are the Yang-Mills vector potentials of the Kaluza-Klein reduction. The Killing vectors satisfy

$$[K^I, K^J] = f^{IJ}{}_K K^K, \quad (2.4)$$

where $f^{IJ}{}_K$ are the structure constants of G . The Yang-Mills field strengths $F^I = \frac{1}{2} F_{\alpha\beta}^I e^\alpha \wedge e^\beta$ are given by

$$F^I = dA^I + \frac{1}{2} f^{IJK} A^J \wedge A^K. \quad (2.5)$$

In an orthonormal basis \hat{e}^A for $d\hat{s}^2$ we find that the Ricci tensor given by

$$\begin{aligned} \hat{R}_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2} K^{Ia} K^J{}_a F^I{}_{\alpha\gamma} F^J{}_\beta{}^\gamma, \\ \hat{R}_{ab} &= R_{ab} + \frac{1}{4} K^I{}_a K^J{}_b F^I{}_{\alpha\beta} F^{J\alpha\beta}, \\ \hat{R}_{ab} &= \hat{R}_{b\alpha} = -\frac{1}{2} K^I{}_b (D_\beta F^I{}_\alpha{}^\beta), \end{aligned} \quad (2.6)$$

where D_γ is the Yang-Mills gauge-covariant derivative,

$$D_\gamma F^I{}_{\alpha\beta} = \nabla_\gamma F^I{}_{\alpha\beta} + f^{IJK} A^I{}_\gamma F^K{}_{\alpha\beta}. \quad (2.7)$$

The curvature scalar is

$$\hat{R} = R_{(5)} + R_{(M)} - \frac{1}{4} K^I{}_a K^{Ja} F^I{}_{\alpha\beta} F^{J\alpha\beta}, \quad (2.8)$$

where $R_{(5)}$ and $R_{(M)}$ are the curvature scalars in spacetime and the internal space M respectively.

⁴We shall adopt the convention throughout this paper of referring to the constant of proportionality Λ in the relation $R_{ab} = \Lambda g_{ab}$ on an Einstein space as the cosmological constant. It sometimes differs by a dimension-dependent factor from other terminologies in the literature, but this one has the merit of simplicity.

The gauge fields also enter in the linearised ansatz for the 5-form field strength [29], as follows:

$$\hat{G}_5 = 4m \epsilon_5 - \frac{1}{m} * F^I \wedge dK^I. \quad (2.9)$$

Substituting (2.6), (2.8) and (2.9) into (2.1) we find that the five-dimensional spacetime components of the ten-dimensional Einstein equation in (2.1) are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R_{(5)} + R_{(M)}) = \frac{1}{2}(F^I{}_{\mu\rho}F^{J\rho}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F^I{}_{\sigma\rho}F^{J\sigma\rho})Y^{IJ}, \quad (2.10)$$

where

$$Y^{IJ} = Y(K^I, K^J) \equiv K^I{}_{m}K^{Jm} + \frac{1}{2m^2}\nabla_m K^I{}_{n}\nabla^m K^{Jn}. \quad (2.11)$$

The possibility of an inconsistency in the Kaluza-Klein reduction becomes apparent from equation (2.10). The left-hand side is independent of the coordinates y on the internal space \mathcal{M}_5 , while the right-hand side is in general y -dependent, since the Killing appearing in Y^{IJ} are in general y -dependent. If the right-hand side does have y -dependence then this is an indication that the assumption that only the *massless* spin-2 field (the five-dimensional spacetime metric) could be retained in the truncation is an invalid one. One can interpret any y -dependence on the right-hand side as indicating that there are bilinear terms, built from the Yang-Mills field strengths, that would act as sources for massive spin-2 fields. Thus it would be inconsistent to make a truncation where the massless gauge bosons are retained, while the massive spin-2 fields are set to zero.

By contrast, this inconsistency problem would be evaded if the quantity Y_{IJ} defined in (2.11) happened to be independent of y . In such a case one could, by taking appropriate linear combinations of the Killing vectors, arrange that

$$Y^{IJ} = \beta \delta^{IJ}, \quad (2.12)$$

where β is a constant. In this circumstance, (2.10) would become precisely the desired five-dimensional Einstein equation, with the right-hand side being the energy-momentum tensor of the Yang-Mills fields.

Remarkably, all the Killing vectors on the round 5-sphere *do* satisfy the condition (2.12), thus providing strong evidence for the probable consistency of the S^5 reduction of type IIB supergravity. On the other hand, it seems that for any other Einstein space \mathcal{M}_5 with positive cosmological constant, the Killing vectors do not in general satisfy the condition (2.12), and thus in these cases a consistent massless truncation in which all the Yang-Mills gauge bosons are retained is not possible.

Note that there is an analogous criterion for the consistency of reductions of $D = 11$ supergravity. This was derived for compactifications on seven-dimensional Einstein spaces in [28], and takes the identical form (2.11) where the Ricci tensor on the internal seven-dimensional space is given by $R_{mn} = 6m^2 g_{mn}$. Similarly, the analogous consistency condition will arise for reductions of $D = 11$ supergravity on four-dimensional internal spaces, and indeed for any of the cases where consistent sphere reductions are known to be possible. A detailed enumeration of these cases is given in [10]. In fact in general one can show that for any round sphere S^n with $R_{mn} = (n-1)m^2 g_{mn}$, the condition (2.12) is satisfied by all the $SO(n+1)$ Killing vectors. (This does not necessarily mean that a consistent Kaluza-Klein reduction on S^n is possible, though.)

It is worth pausing here to emphasise that although we have derived the consistency condition that (2.11) must be constant by means of a consideration only of the *linearised* ansatz for the Kaluza-Klein reduction of the gauged fields, the result is a completely general one. The reason for this is discussed in detail in [26]). The crucial point is the following. If (2.11) is y -dependent, this shows that in a complete Kaluza-Klein reduction in which all the massive as well as massless fields were retained, there would be trilinear couplings involving one power of a heavy spin-2 field, say $H_{\mu\nu}^{IJ}$, coupling to the bilinear source term quadratic in gauge fields $F_{\mu\rho}^I$ on the right-hand side of (2.10):

$$\mathcal{L}_{\text{int}} = H^{IJ\mu\nu} (F_{\mu\rho}^I F_{\nu}^{J\rho} - \frac{1}{4} g_{\mu\nu} F_{\sigma\rho}^I F^{J\sigma\rho}). \quad (2.13)$$

Now, the masses of all the lower-dimensional massive fields are acquired through a Higgs mechanism, and so it follows that the original gauge invariances must remain unbroken. In consequence, the lower-dimensional massive spin-2 field $H_{\mu\nu}^{IJ}$ must have a gauge invariance, implying that the source-current that couples to it must be conserved [26]. Indeed, from an order-by-order analysis it follows that the bilinear current must be conserved by virtue of the free field equations. The bilinear current

$$(F_{\mu\rho}^I F_{\nu}^{J\rho} - \frac{1}{4} g_{\mu\nu} F_{\sigma\rho}^I F^{J\sigma\rho}) \quad (2.14)$$

appearing in (2.13) is the *unique* one with this property, and so it is not possible for it to receive any corrections as a result of including higher-order non-linear terms in the reduction ansatz. Thus there is no possibility that the inconsistency we are highlighting could “disappear” in a more complete higher-order analysis. If the quantity (2.11) turns out to be y -dependent, then no consistent Kaluza-Klein reduction in which the associated gauge fields are retained is possible.

In order to proceed with our discussion, it is now necessary to study in detail the Killing vectors on the internal space. We shall give a rather general discussion, which encompasses many of the compactifications of type IIB supergravity and eleven-dimensional supergravity as special cases. Later, in section 5, we shall apply these results to study the consistency of the Kaluza-Klein reductions.

3 Construction of Killing vectors on the internal space

3.1 Killing vectors on $U(1)$ bundles

Consider a D -dimensional manifold with a group G of isometries. Suppose that there exists a $U(1)$ connection on M whose curvature is invariant under the isometry group G . One then expects that the natural metric on the $(D+1)$ -dimensional bundle space with $U(1)$ fibers corresponding to this G -invariant $U(1)$ connection should contain $G \times U(1)$ in its isometry group.⁵

To see that this is indeed the case, suppose that the metric on the base manifold is ds^2 , and that the invariant $U(1)$ connection is A . The natural metric on the $(n+1)$ -dimensional bundle space is then taken to be

$$d\hat{s}^2 = c^2(dz - A)^2 + ds^2, \quad (3.1)$$

where c is a constant, and z is the coordinate on the $U(1)$ fibre. (From this point on, we adopt the convention that quantities with hats refer to the total bundle space \hat{M} , while quantities without hats refer to the base space M . We shall use indices m, n, \dots in the total bundle space \hat{M} , and indices a, b, \dots in the base space M .) In the obvious orthonormal frame, the Riemann tensor for $d\hat{s}^2$ has components

$$\begin{aligned} \hat{R}_{abcd} &= R_{abcd} - \frac{1}{4}c^2(F_{ac}F_{bd} - F_{ad}F_{bc} + 2F_{ab}F_{cd}), \\ \hat{R}_{zazb} &= \frac{1}{4}c^2F_a^cF_{bc}, \\ \hat{R}_{abcz} &= \frac{1}{2}c\nabla_cF_{ab}, \end{aligned} \quad (3.2)$$

where R_{abcd} is the Riemann tensor of the metric ds^2 on the base manifold.

⁵Generically, this will be the full isometry group of the bundle space, but in special cases it could be a larger group containing $G \times U(1)$ as a subgroup. An example is when the base manifold is CP^n , and the bundle space is the sphere S^{2n+1} . If the length of the $U(1)$ fibres is chosen so as to give the “round” sphere, then the generic $SU(n+1) \times U(1)$ isometry group in the bundle enlarges to $SO(2n+2)$.

From (3.2) it follows that the components of the Ricci tensor for $d\hat{s}^2$ are

$$\begin{aligned}\hat{R}_{ab} &= R_{ab} - \frac{1}{2}c^2 F_{ac} F_b{}^c, \\ \hat{R}_{zz} &= \frac{1}{4}c^2 F^{ab} F_{ab}, \\ \hat{R}_{az} &= -\frac{1}{2}c \nabla^b F_{ab},\end{aligned}\tag{3.3}$$

where R_{ab} is the Ricci tensor on the base manifold.

We now make the following ansatz in order to lift a Killing vector K on the base manifold M to a Killing vector \hat{K} on the total bundle space \hat{M} :

$$\hat{K} = K + h \partial_z, \tag{3.4}$$

where h is a function to be determined. By substituting our ansatz (3.4) into the Killing equation of the bundle space:

$$\hat{\nabla}_m \hat{K}_n + \hat{\nabla}_n \hat{K}_m = 0, \tag{3.5}$$

we find that \hat{K} is a Killing vector on the bundle space provided that h satisfies the following two equations:

$$\partial_a h = \mathcal{L}_K A_a, \tag{3.6}$$

$$\partial_z h = 0, \tag{3.7}$$

where $\mathcal{L}_K A_a$ is the Lie derivative, defined by

$$\mathcal{L}_K A_a \equiv K^b \partial_b A_a + A_b \partial_a K^b = K^b \nabla_b A_a + A_b \nabla_a K^b. \tag{3.8}$$

Equation (3.6) can be rewritten in terms of the field strength $F = dA$ as

$$\partial_a h = K^b F_{ba} + \partial_a (K^b A_b). \tag{3.9}$$

It is easy to see that the two equations (3.9) and (3.7) always admit a solution, provided that F is invariant under the action of the Killing symmetry generated by K . Clearly (3.7) is nothing more than the statement that h is independent of the fibre coordinate z . The integrability condition for solving (3.9) for h is that the right-hand side should be expressible as the gradient of a scalar. Since the second term is already a gradient, this means that we must just show that $\nabla_c (K^b F_{ba}) - \nabla_a (K^b F_{bc}) = 0$. Calculating this expression, we find

$$\nabla_c (K^b F_{ba}) - \nabla_a (K^b F_{bc}) = -K^b \nabla_b F_{ac} - F_{ab} \nabla_c K^b - F_{bc} \nabla_a K^b, \tag{3.10}$$

which is nothing but the Lie derivative $\mathcal{L}_K F_{ca}$. This vanishes precisely by virtue of the assumption that F is invariant under the Killing symmetry.

The above argument establishes that every Killing vector on the base manifold lifts to one in the total bundle space. In addition to these Killing vectors of the isometry group G of the base manifold, there will also be the $U(1)$ Killing vector $\partial/\partial z$ on the $U(1)$ fibres. Thus the isometry group of the total bundle space will be at least $G \times U(1)$.

We are interested in obtaining an explicit construction of the Killing vectors in certain Einstein spaces that can be used for Kaluza-Klein reduction, in order to test the consistency as described in section 2. In all the examples that we shall consider, the Einstein space can be constructed as a $U(1)$ bundle over a Kähler base manifold. More specifically, in all cases of interest this Kähler space will itself be a direct product of Einstein-Kähler spaces. The additional structure of the Kähler spaces allows us to obtain more explicit constructions for the Killing vectors in the bundle space.

3.2 Killing vectors on Kähler spaces and their $U(1)$ bundle spaces

We begin with a review of some basic properties of Killing vectors and Kähler spaces. Consider a compact Kähler manifold M equipped with a positive definite metric g_{ab} , and a Kähler form J_{ab} . We are interested in the case where M has continuous isometries, and hence admits Killing vectors. It follows from the defining equation $\nabla_a K_b + \nabla_b K_a = 0$ for a Killing vector that

$$\square K_a + R_{ab} K^b = 0. \quad (3.11)$$

Multiplying by K^a , integrating over M , and integrating by parts, gives

$$\int_M (-|\nabla_a K_b|^2 + R_{ab} K^a K^b) = 0, \quad (3.12)$$

where $|\nabla_a K_b|^2$ means $(\nabla_a K_b)(\nabla^a K^b)$. The metric is positive-definite, and so from (3.12) we deduce that for Killing vectors to exist, there must be appropriate non-negative contributions from the Ricci-tensor term. In fact we are interested in the case where R_{ab} is positive definite.

Another consequence that follows from the positivity of the Ricci tensor is that the first Betti number b_1 of the Kähler space must be zero. This follows from an argument precisely paralleling the one above concerned with the possibility of the existence of Killing vectors. A harmonic 1-form H_a satisfies the equation $-\square H_a + R_{ab} H^b = 0$. By multiplying by H^a , integrating over M , and integrating by parts on the first term, we see that there can be no harmonic 1-forms if the Ricci tensor is positive definite. Since we shall be considering spaces that have strictly positive-definite Ricci tensors, it follows that they will have $b_1 = 0$, and admit no harmonic 1-forms.

Consider now the vector V^a , constructed from the Killing vector K^a as follows:

$$V^a \equiv J^a{}_b K^b. \quad (3.13)$$

Our first goal will be to show that V^a can be written as the gradient of a scalar function. To prove this, define

$$Q_{ab} \equiv \nabla_a V_b - \nabla_b V_a. \quad (3.14)$$

It follows that

$$\begin{aligned} |Q_{ab}|^2 &= (\nabla_a V_b - \nabla_b V_a)(\nabla^a V^b - \nabla^b V^a), \\ &= 2(\nabla_a K_b)(\nabla^a K^b) - 2J^{ad} J^{cb} (\nabla_a K_b)(\nabla_c K_d). \end{aligned} \quad (3.15)$$

Integrating this over M , and integrating by parts on each term, we obtain

$$\begin{aligned} \int_M |Q_{ab}|^2 &= -2 \int_M K^a \square K_a + 2 \int_M J^{ad} J^{cb} K_b \nabla_a \nabla_c K_d, \\ &= -2 \int_M K^a \square K_a + 2 \int_M J^{ad} J^{cb} K_b R^e{}_{acd} K_e, \\ &= -2 \int_M K^a (\square K_a + R_{ab} K^b), \\ &= 0, \end{aligned} \quad (3.16)$$

where we have used the standard Killing-vector identity $\nabla_a \nabla_c K_d = R^e{}_{acd} K_e$ in reaching the second line, and the standard Kähler identity $R_{abcd} = J_c{}^e J_d{}^f R_{abef}$ in reaching the third line. The final result follows from using (3.11). Thus we conclude that $Q_{ab} = 0$, and hence that V_a , viewed as a 1-form, is closed; $dV = 0$. Locally, therefore, we can write $V = -d\psi$. As we discussed previously, we shall be interested in Kähler spaces with positive-definite Ricci tensor, and such spaces have vanishing first Betti number. Since there are no harmonic 1-forms in such spaces, it follows that $dV = 0$ can be solved globally by writing $V = -d\psi$. In other words, we have the result that on a Kähler space with vanishing first Betti number, any Killing vector can be written as

$$K^a = J^{ab} \partial_b \psi, \quad (3.17)$$

for some scalar ψ .

This scalar ψ has a clear interpretation if we impose that our Kähler space is also an Einstein space, $R_{ab} = \Lambda g_{ab}$, where Λ is the “cosmological constant” on M . Then (3.11) reduces to

$$\square K_a + \Lambda K_a = 0. \quad (3.18)$$

It is now straightforward to see, by substituting (3.17) into (3.18), that this scalar field is actually an eigenfunction of the Laplacian on M , $-\square\psi = \lambda\psi$, with

$$\square\psi + 2\Lambda\psi = 0. \quad (3.19)$$

Moreover, the implication goes in the other direction as well. In other words, if ψ is an eigenfunction of the scalar Laplacian, satisfying (3.19), then K^a defined by (3.17) is a Killing vector. To see this, we define $P_{ab} \equiv \nabla_a K_b + \nabla_b K_a$. Substituting (3.17) into this, writing $\int |P|^2$, and then performing appropriate integrations by parts, we find that

$$\int_M |P_{ab}|^2 = 2(\lambda - 2\Lambda) \int_M |\nabla_a \psi|^2. \quad (3.20)$$

(Again, standard results from Kähler geometry are needed in intermediate steps.) Thus if the scalar eigenfunction ψ has eigenvalue $\lambda = 2\Lambda$, it follows that $P_{ab} = 0$ and hence that K_a constructed as in (3.17) is a Killing vector.

Thus we see that there is a one-to-one correspondence between Killing vectors, and scalar eigenfunctions with eigenvalue $\lambda = 2\Lambda$, where Λ is the cosmological constant of the Einstein-Kähler space.

Using (3.17), we can now obtain explicit expressions for the Killing vectors on the space of the $U(1)$ bundle over an Einstein-Kähler base space, where the curvature of the $U(1)$ connection is taken to be proportional to the Kähler form. Taking the Einstein-Kähler base metric to have cosmological constant Λ as above, and taking the field strength of the connection A on the $U(1)$ bundle to be $F = \alpha J$, where α is a constant, it follows from (3.3) that the Ricci tensor on the bundle space will be given by

$$\hat{R}_{ab} = (\Lambda - \frac{1}{2}c^2\alpha^2)\delta_{ab}, \quad \hat{R}_{zz} = \frac{1}{4}c^2\alpha^2 D, \quad \hat{R}_{az} = 0, \quad (3.21)$$

where D is the dimension of the base manifold. In particular, the metric on the $U(1)$ bundle becomes Einstein if a is chosen such that

$$\Lambda = \frac{1}{4}c^2\alpha^2(D + 2). \quad (3.22)$$

Substituting (3.17) into (3.9), we now obtain the result that

$$\partial_a h = \partial_a (\alpha\psi + K^b A_b), \quad (3.23)$$

which can be integrated to give $h = \alpha\psi + K^b A_b$. Thus for each Killing vector K on the Einstein-Kähler base space, with its associated scalar ψ as given in (3.17), the corresponding Killing vector in the $U(1)$ bundle space is

$$\hat{K} = K + (\alpha\psi + K^b A_b) \frac{\partial}{\partial z}. \quad (3.24)$$

It is worth noting at this stage that there is an elegant expression for the Killing vector \hat{K} , viewed as a 1-form by lowering its vector index using the metric (3.1) on the bundle space. After doing this, we find that as a 1-form we have

$$\hat{K} = -i(\partial - \bar{\partial})\psi + \alpha c^2 \psi (dz - A), \quad (3.25)$$

where ∂ and $\bar{\partial}$ are the holomorphic and anti-holomorphic exterior derivatives; $d = \partial + \bar{\partial} = d\zeta_\alpha \partial_\alpha + d\bar{\zeta}^{\bar{\alpha}} \partial_{\bar{\alpha}}$.

Note that in the case of an Einstein-Kähler base space we can easily express ψ in terms of the Killing vector K^a , since from (3.17) we have $J^{ab} \partial_a K_b = -\square \psi$, and hence from (3.19) we shall have

$$\psi = \frac{1}{2\Lambda} J^{ab} \partial_a K_b. \quad (3.26)$$

3.3 Killing vectors on a product of Kähler spaces and their $U(1)$ bundles

In subsequent sections, we shall be interested in constructing Killing vectors on $U(1)$ bundles over products of 2-spheres and more generally complex projective spaces CP^n . These are particular examples of Einstein-Kähler spaces. In this section, we shall give results for the construction of Killing vectors on $U(1)$ bundles over the direct product of N Einstein-Kähler spaces M_i , of real dimensions d_i , *i.e.* $M = M_1 \times M_2 \times \dots \times M_N$, with total real dimension

$$D = \sum_{i=1}^N d_i. \quad (3.27)$$

The metric on the bundle space will be given by

$$d\hat{s}^2 = c^2 (dz - A)^2 + \sum_{i=1}^N ds_i^2, \quad (3.28)$$

where ds_i^2 is the Einstein-Kähler metric on the factor M_i in the base space, with cosmological constant Λ_i . The total connection A is equal to the sum of contributions from each factor, $A = \sum_i A^{(i)}$.

Since the base space is a direct product, we can choose the natural block-diagonal basis for its Killing vectors, where there is no mixing between the isometries of each factor in the product. Thus if $K^{(i)}$ is a Killing vector on M_i then it is also a Killing vector on M , and *vice versa*. If we use a_i to denote a coordinate index on M_i , then this result follows by combining the Killing equation $\nabla_{a_i} K_{b_i} + \nabla_{b_i} K_{a_i} = 0$ on M_i , with the fact that the $K^{(i)}$ are covariantly constant with respect to ∇_{b_j} for $j \neq i$.

This fact allows us to use the results of the previous section to express any of these Killing vectors $K^{(i)}$ on M as

$$K^{a_i} = J^{a_i b_i} \partial_{b_i} \psi^{(i)}, \quad (3.29)$$

where $\psi^{(i)}$ is the corresponding scalar eigenfunction of the Laplacian on M_i with eigenvalue $2\Lambda_i$, and $J_{a_i b_i}$ are the components of the Kähler form on M_i . From the results in the previous section, it then follows that the corresponding Killing vector in the bundle space will be

$$\hat{K}^{(i)} = K^{(i)} + (\alpha_i \psi^{(i)} + K^{b_i} A_{b_i}) \frac{\partial}{\partial z}, \quad (3.30)$$

where A_{a_i} is the contribution to A from the factor M_i in the base space, $A = \sum_i A^{(i)}$, and we are taking

$$F = dA = \sum_i \alpha_i J^{(i)}, \quad (3.31)$$

where $J^{(i)}$ is the Kähler form on M_i .

We may again obtain an elegant expression for the Killing vector viewed as a 1-form, generalising (3.25):

$$\hat{K}^{(i)} = -i(\partial - \bar{\partial}) \psi^{(i)} + \alpha_i c^2 \psi^{(i)} (dz - A). \quad (3.32)$$

The period Δz of the fibre coordinate z must be compatible with the integrals of F over all 2-cycles in the base manifold. Specifically, we must have

$$\Delta z = \frac{1}{q_k} \int_{\Sigma_k} F, \quad (3.33)$$

where q_k is an integer and Σ_k is any 2-cycle in the base space. We are taking each factor M_i in the base space to be Einstein-Kähler, with cosmological constant Λ_i , and so it follows that the Ricci form $P^{(i)}$ in M_i is given by $P^{(i)} = \Lambda_i J^{(i)}$. Since $1/(2\pi) P^{(i)}$ defines the first Chern class of M_i , it follows that $1/(2\pi) \int P^{(i)} = \text{integer}$, where the integral is taken over any 2-cycle in M_i , whilst the integral will be zero for any 2-cycle in M_j with $j \neq i$. If we define k_i to be the greatest common divisor of the integers obtained by integrating $P^{(i)}$ over all possible 2-cycles in M_i , then it follows from (3.33) that z must have a period such that

$$\Delta z = \frac{2\pi \alpha_i k_i}{\Lambda_i q_i}, \quad (3.34)$$

for all i , where the q_i are integers. Thus we must have

$$\alpha_i = \frac{b \Lambda_i q_i}{k_i}, \quad (3.35)$$

where b is related to the period Δz by $b = \Delta z/(2\pi)$, and it is a constant independent of i . Since we have also included the constant c in (3.1), we are free to choose b at will, to give z

a convenient period. The integers q_i can be thought of as the winding numbers of the $U(1)$ bundle over each factor M_i in the product base manifold.

Note that if one wants the total $U(1)$ bundle space to be Einstein, with cosmological constant $\hat{\Lambda}$, then it follows from (3.3) that we must have

$$\hat{\Lambda} = \Lambda_i - \frac{1}{2}c^2 \alpha_i^2, \quad \text{for all } i, \quad (3.36)$$

$$\hat{\Lambda} = \frac{1}{4}c^2 \sum_i d_i \alpha_i^2, \quad (3.37)$$

where d_i is the dimension of the manifold M_i , and α_i is given by (3.35). To solve these equations, one can view $\hat{\Lambda}$ and the winding numbers q_i as freely-specifiable quantities, with the N equations (3.36) then being solved for the individual cosmological constants Λ_i of the factors in the base space, and (3.37) being solved for the scale factor c in fibre direction of the metric (3.28) on the bundle space. As we shall now show, one can always solve these equations for the Λ_i and c , for any choice of the integers q_i , provided that they are not all zero.⁶

To see this, we substitute (3.35) into (3.36), and note that for each i the equation allows a real solution for Λ_i only if

$$c^2 \alpha_i^2 \leq \frac{\Lambda_i^2}{2\hat{\Lambda}}. \quad (3.38)$$

Summing over i , and using (3.37) then gives

$$\hat{\Lambda}^2 \leq \frac{1}{8} \sum_i d_i \Lambda_i^2. \quad (3.39)$$

On the other hand, combining (3.36) and (3.37) we have

$$\hat{\Lambda} = \frac{1}{D+2} \sum_i d_i \Lambda_i. \quad (3.40)$$

Combining (3.39) and (3.40) then gives the result

$$(D+2)^2 \sum_i d_i \Lambda_i^2 - 8 \sum_{i,j} d_i d_j \Lambda_i \Lambda_j \geq 0. \quad (3.41)$$

This is the criterion for the existence of a real Einstein space. Since it is just a quadratic form in Λ_i , it can be expressed as the condition that the $N \times N$ matrix M_{ij} , defined by

$$M_{ij} = (D+2) d_i \delta_{ij} - 8 d_i d_j, \quad (3.42)$$

⁶If some of the q_i are zero, we can just separate off the corresponding Einstein spaces M_i in the base space, and prove the existence of an Einstein metric on the bundle over the remaining base-space factors for which all the q_i are non-zero. The product of this bundle space with the Einstein spaces associated with the $q_i = 0$ factors can clearly be made Einstein, by appropriate choice of the Λ_i . If *all* the q_i were zero the $U(1)$ bundle would be trivial and the total $(D+1)$ -dimensional space would be $S^1 \times M$, which clearly cannot be Einstein since the factors in the base space M are assumed to have strictly-positive cosmological constants.

must have non-negative eigenvalues.

To show this, we first note that the matrix M_{ij} has determinant given by

$$\det(M_{ij}) = (D-2)^2 (D+2)^{2N-2} \prod_i d_i, \quad (3.43)$$

which is strictly positive, since we may always assume $D > 2$.⁷

Secondly, we note that if the dimensions d_i are all taken to be equal, $d_i = d$, then the eigenvalues of M_{ij} are $(D+2)^2 d$ (occurring $N-1$ times) and $(D-2)^2 d$ (occurring once). Thus in this special case all the eigenvalues of M_{ij} are strictly positive. If M_{ij} were to have any negative values for any valid choice of the d_i , it would have to be the case that $\det(M_{ij})$ passed through 0 as the parameters d_i were adjusted from $d_i = d$ to these putative values of d_i . However, we saw from (3.43) that the determinant is strictly positive, and so it follows that M_{ij} cannot have negative eigenvalues for any valid choice of d_i . Thus it is guaranteed that the inequality (3.41) is satisfied, and so a real solution to the conditions (3.36) and (3.37) always exists.

Although we have given an existence proof for an Einstein metric on the bundle spaces for any choice of the winding numbers q_i , it is not in general easy to solve explicitly for the cosmological constants Λ_i of the individual factors in the base space. (In general, one has to solve high-order polynomial equations.) However, a simple solution of (3.36) and (3.37) can always be explicitly obtained in the special case where we choose the winding numbers q_i to be such that $q_i = k_i/\ell$, where $\ell = \gcd(k_i)$ is the greatest common divisor of the k_i . In this case, from (3.36) we see that this set of N equations, labelled by i , all become equivalent. Therefore, defining $\Lambda \equiv \Lambda_i$ and $\alpha \equiv \alpha_i$, we have

$$\Lambda = \frac{D+2}{D} \hat{\Lambda}, \quad (3.44)$$

and

$$c^2 \alpha^2 = \frac{4}{D+2} \Lambda, \quad (3.45)$$

where D is the total dimension of the base manifold. Combining (3.35) and (3.45) it follows that the parameters of the metric satisfy the relation

$$\Lambda b^2 c^2 = \frac{4\ell^2}{D+2}. \quad (3.46)$$

Note that since in this special case we have all the Λ_i equal, the product of Einstein-Kähler base spaces is itself an Einstein space. This situation with $q_i = k_i/\ell$ will be seen to be of

⁷The case where the total dimension D of the base space is equal to 2 can easily be disposed of in a separate discussion. The only possibility would be for the base space to be S^2 , and we already know that the $U(1)$ bundle over this is S^3 , which admits an Einstein metric.

particular significance in the next section, when we take the factors in the product base space all to be complex projective spaces. It turns out that the Einstein spaces with $q_i = k_i/\ell$ then all admit 2 Killing spinors.

4 Products of CP^n spaces, and their $U(1)$ bundles

4.1 Geometry of CP^n , and its Killing vectors

We begin by reviewing the Fubini-Study construction of the Einstein-Kähler metric on CP^n . Let Z^A be complex coordinates on C^{n+1} , with the flat metric

$$ds_{2n+2}^2 = dZ^A d\bar{Z}_A. \quad (4.1)$$

We shall split the index A into $A = (0, \alpha)$, where $1 \leq \alpha \leq n$, and introduce inhomogeneous coordinates $\zeta^\alpha = Z^\alpha/Z^0$, in the patch where $Z^0 \neq 0$. We make the further definitions

$$Z^0 = e^{i\tau} |Z^0|, \quad r = \sqrt{Z^A \bar{Z}_A}, \quad f = 1 + \zeta^\alpha \bar{\zeta}^\alpha. \quad (4.2)$$

Substituting into (4.1), we find that the flat metric on C^{n+1} becomes

$$ds_{2n+2}^2 = dr^2 + r^2 d\Omega_{2n+1}^2, \quad (4.3)$$

where $d\Omega_{2n+1}^2$ is the metric on the unit sphere S^{2n+1} , given by

$$d\Omega_{2n+1}^2 = (d\tau + B)^2 + f^{-1} d\zeta^\alpha d\bar{\zeta}^\alpha - f^{-2} \bar{\zeta}^\alpha \zeta^\beta d\zeta^\alpha d\bar{\zeta}^\beta, \quad (4.4)$$

where

$$B = \frac{1}{2} i f^{-1} (\zeta^\alpha d\bar{\zeta}^\alpha - \bar{\zeta}^\alpha d\zeta^\alpha). \quad (4.5)$$

The metric (4.4) is the unit S^{2n+1} described as a $U(1)$ bundle over CP^n , and the last two terms are precisely the Fubini-Study metric $d\Sigma_n^2$ on CP^n :

$$d\Sigma_n^2 = f^{-1} d\zeta^\alpha d\bar{\zeta}^\alpha - f^{-2} \bar{\zeta}^\alpha \zeta^\beta d\zeta^\alpha d\bar{\zeta}^\beta, \quad (4.6)$$

and so

$$d\Omega_{2n+1}^2 = (d\tau + B)^2 + d\Sigma_n^2. \quad (4.7)$$

The quantity B defined in (4.5) is a potential for the Kähler form, with

$$J = dB = i f^{-1} d\zeta^\alpha \wedge d\bar{\zeta}^\alpha + i f^{-2} \bar{\zeta}^\alpha \zeta^\beta d\zeta^\alpha \wedge d\bar{\zeta}^\beta, \quad (4.8)$$

which is the Kähler form. This can be written as $J = i g_{\alpha\bar{\beta}} d\zeta^\alpha \wedge d\bar{\zeta}^\beta$, where the metric $g_{\alpha\bar{\beta}}$ and its inverse $g^{\alpha\bar{\beta}}$ are given by

$$g_{\alpha\bar{\beta}} = \frac{1}{2} f^{-1} \delta_{\alpha\bar{\beta}} - \frac{1}{2} f^{-2} \bar{\zeta}^\alpha \zeta^\beta, \quad g^{\alpha\bar{\beta}} = 2f \delta^{\alpha\bar{\beta}} + 2f \zeta^\alpha \bar{\zeta}^\beta. \quad (4.9)$$

The Fubini-Study metric (4.6) is Einstein, with cosmological constant

$$\Lambda = 2(n+1). \quad (4.10)$$

We shall refer to the Fubini-Study metric (4.6) with this specific normalisation for the cosmological constant as the “unit CP^n metric,” since it is the one that corresponds to the Hopf fibration of the unit $(2n+1)$ -sphere. Note that CP^n has the isometry group $SU(n+1)$, which can be seen from the fact that the metric (4.1) and the coordinate r are both invariant under $SU(n+1)$, acting by matrix multiplication on the column vector Z^A .

Since we eventually want to be able to construct Killing vectors on $U(1)$ bundles over products of CP^n spaces, we need to find the eigenfunctions of the scalar Laplacian on CP^n with eigenvalue 2Λ , as discussed in section (3.2). In fact the construction of all scalar eigenfunctions on CP^n is very simple. Let $T_{A_1 \dots A_p}{}^{B_1 \dots B_q}$ be a constant Hermitean $SU(n+1)$ tensor, which is symmetric in the index set $\{A_1, \dots, A_p\}$ and the index set $\{B_1, \dots, B_q\}$, and traceless in any contraction between an A and a B index. This defines the (p, q) representation of $SU(n+1)$. Clearly the scalar function

$$\Phi = T_{A_1 \dots A_p}{}^{B_1 \dots B_q} Z^{A_1} \dots Z^{A_p} \bar{Z}_{B_1} \dots \bar{Z}_{B_q} \quad (4.11)$$

is a zero mode of the Laplacian on C^{n+1} :

$$\square_{C^{n+1}} \Phi = \frac{\partial^2}{\partial Z^A \partial \bar{Z}_A} \Phi = 0, \quad (4.12)$$

where we can write this Laplacian in terms of r and the Laplacian on the unit S^{2n+1} as

$$0 = \square_{C^{n+1}} \Phi = \frac{1}{r^{2n+1}} \frac{\partial}{\partial r} \left(r^{2n+1} \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \square_{S^{2n+1}} \Phi. \quad (4.13)$$

Note that Φ can be written as

$$\Phi = r^{p+q} e^{i(p-q)\tau} \Psi, \quad (4.14)$$

where Ψ depends only on the inhomogeneous CP^n coordinates ζ^α .

It is straightforward to show from (4.7) that the components of the sphere metric \hat{g}_{AB} and the CP^n metric g_{ab} are related by

$$\begin{aligned} \hat{g}_{ab} &= g_{ab} + B_a B_b, & \hat{g}_{a\tau} &= B_a, & \hat{g}_{\tau\tau} &= 1, \\ \hat{g}^{ab} &= g^{ab}, & \hat{g}^{a\tau} &= -B^a, & \hat{g}^{\tau\tau} &= 1 + B_a B^a, \end{aligned} \quad (4.15)$$

where $B^a \equiv g^{ab} B_b$. From this it is easily seen that the scalar Laplacian on S^{2n+1} is given by

$$\square_{S^{2n+1}} = \left(\nabla_a - B_a \frac{\partial}{\partial \tau} \right)^2 + \frac{\partial^2}{\partial \tau^2}. \quad (4.16)$$

Substituting (4.14) into (4.13) and (4.16), we therefore find that Ψ is an eigenfunction on CP^n , satisfying

$$-\mathcal{D}_a \mathcal{D}^a \Psi = 2[2p q + n(p + q)] \Psi, \quad (4.17)$$

where $\mathcal{D}_a = \nabla_a - i(p - q) B_a$. This is the Laplacian for scalar fields of charge $(p - q)$, in the $(p, 0, 0, \dots, 0, q)$ representation of $SU(n + 1)$. The uncharged scalars therefore occur in the $(p, 0, 0, \dots, 0, p)$ representations, with eigenvalues $\lambda = 4p(p + n)$.

In section (3.2) the Killing vectors on an Einstein-Kähler space were constructed in terms of uncharged scalar eigenfunctions with eigenvalue 2Λ . On CP^n , the appropriate eigenfunctions are the ones with $(p, q) = (1, 1)$, since, as can be seen from (4.17), they have eigenvalue $4(n + 1)$, which, from (4.10), is 2Λ . They are indeed in the adjoint representation of $SU(n + 1)$, as should be since they are supposed to be in one-to-one correspondence with the Killing vectors of CP^n .

Thus we see that the scalars ψ that generate the Killing vectors on CP^n are given by

$$\psi = \frac{1}{r^2} T_A{}^B Z^A \bar{Z}_B, \quad (4.18)$$

where $T_A{}^B$ is an arbitrary Hermitean traceless tensor. From the previous definitions, it has the following expression in terms of the inhomogeneous coordinates on CP^n :

$$\psi = f^{-1} (T_0{}^0 + T_0{}^\alpha \bar{\zeta}^{\bar{\alpha}} + T_\alpha{}^0 \zeta^\alpha + T_\alpha{}^\beta \zeta^\alpha \bar{\zeta}^{\bar{\beta}}). \quad (4.19)$$

Note that since $T_A{}^B$ is traceless, we can write $T_0{}^0 = -T_\alpha{}^\alpha$, and thus we can regard the unconstrained constant tensors $T_0{}^\alpha$, $T_\alpha{}^0$ and $T_\alpha{}^\beta$ as parameterising the set of scalars ψ corresponding to the full set of $n(n + 2)$ Killing vectors of CP^n .

From the scalars ψ , we can readily construct the Killing vectors using (3.17). From (4.9) we therefore find that the complex components of the Killing vector associated with ψ are given by

$$K^\alpha = i g^{\alpha\bar{\beta}} \partial_{\bar{\beta}} \psi = \frac{i}{2} (T_0{}^\alpha + T_\beta{}^\alpha \zeta^\beta - T_0{}^0 \zeta^\alpha - T_\beta{}^0 \zeta^\beta \zeta^\alpha), \quad (4.20)$$

with $K^{\bar{\alpha}}$ being the complex conjugate of K^α .

As a check on this construction of the Killing vectors from the scalar eigenfunctions ψ , we may also construct them directly, using the fact that they must correspond to infinitesimal $SU(n + 1)$ transformations of the form $\delta Z^A = i\epsilon T_B{}^A Z^B$ on the homogeneous coordinates,

where T_B^A is again an arbitrary Hermitean traceless tensor. This translates into $\delta\zeta^\alpha = \delta Z^\alpha/Z^0 - Z^\alpha/(Z^0)^2 \delta Z^0$, giving

$$\delta\zeta^\alpha = i\epsilon (T_0^\alpha + T_\beta^\alpha \zeta^\beta - T_0^0 \zeta^\alpha - T_\beta^0 \zeta^\beta \zeta^\alpha), \quad (4.21)$$

which is in precise agreement with (4.20), since Killing vectors generate the coordinate transformations $\delta\zeta^\alpha = 2\epsilon K^\alpha$. Of course we also need to know the explicit scalar functions ψ , for the purpose of lifting the Killing vectors to the $U(1)$ bundle space.

Note that CP^n is a space of constant holomorphic sectional curvature, and in fact in terms of a real index notation the orthonormal components of the Riemann tensor of the unit CP^n with metric (4.6) are given by

$$R_{abcd} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} + J_{ac} J_{bd} - J_{ad} J_{bc} + 2J_{ab} J_{cd}. \quad (4.22)$$

It is sometimes useful to work with an explicit real metric for CP^n . In Appendix A, we obtain an iterative construction for a real metric on CP^n , in terms of a metric on CP^{n-1} .

It is now straightforward to follow the procedure described in sections (3.2) and (3.3), to construct the $U(1)$ bundle space over an arbitrary product of CP^n metrics. Specifically, we take the base manifold to be $M = M_1 \times M_2 \times \dots \times M_N$, where M_i is the complex projective space CP^{n_i} , with real dimension $d_i = 2n_i$. We shall denote the total bundle spaces by

$$Q_{n_1 n_2 \dots n_N}^{q_1 q_2 \dots q_N}, \quad (4.23)$$

where the integers q_i are the winding numbers of the $U(1)$ bundle over the factors CP^{n_i} in the base manifold.

4.2 Killing spinors on $Q_{n_1 \dots n_N}^{q_1 \dots q_N}$ spaces

As we discussed in section 3.3, one can always find a solution to the conditions (3.36) and (3.37) for any choice of the q_i . A particularly simple case is when $q_i = k_i$. In fact in CP^{n_i} there is only one 2-cycle, and the integer k_i is therefore simply the result from integrating the first Chern class $P_i/(2\pi)$ over this cycle, which turns out to give

$$k_i = n_i + 1. \quad (4.24)$$

In fact the Einstein spaces $Q_{n_1 n_2 \dots n_N}^{q_1 q_2 \dots q_N}$ with $q_i = (n_i + 1)/\ell$ where ℓ is the greatest common divisor of the $(n_i + 1)$ have a further nice feature, namely that they all admit Killing spinors. To show this, we note that the Killing spinor equation

$$D_A \eta - \frac{i}{2} \sqrt{\frac{\hat{\Lambda}}{D}} \Gamma_A \eta = 0 \quad (4.25)$$

has the integrability condition

$$\frac{1}{4}\hat{R}_{ABCD}\Gamma^{CD}\eta - \frac{\hat{\Lambda}}{2D}\Gamma_{AB}\eta = 0, \quad (4.26)$$

which is obtained by taking a commutator of the generalised derivatives appearing in (4.25). From (4.26) one can easily deduce that the metric on the total bundle space must be Einstein, and furthermore that

$$\hat{C}_{ABCD}\Gamma^{CD}\eta = 0, \quad (4.27)$$

where \hat{C}_{ABCD} is the Weyl tensor on the total space.

If for every space CP^{n_i} we take $q_i = k_i/\ell$, where $\ell = \gcd(k_i)$, then we can use (3.44) and (3.45) to express the non-zero orthonormal components of the Riemann tensor on the $U(1)$ bundle space as:

$$\begin{aligned} \hat{R}_{a_i b_i c_i d_i} &= \frac{\Lambda}{d_i + 2} \left(\delta_{a_i c_i} \delta_{b_i d_i} - \delta_{a_i d_i} \delta_{b_i c_i} \right) \\ &\quad + \Lambda \left[\frac{1}{d_i + 2} - \frac{1}{D + 2} \right] (J_{a_i c_i} J_{b_i d_i} - J_{a_i d_i} J_{b_i c_i} + 2 J_{a_i b_i} J_{c_i d_i}), \\ \hat{R}_{a_i b_i a_j b_j} &= -\frac{2\Lambda}{D + 2} J_{a_i b_i} J_{a_j b_j}, \\ \hat{R}_{a_i a_j b_i b_j} &= -\frac{\Lambda}{D + 2} J_{a_i b_i} J_{a_j b_j}, \\ \hat{R}_{0 a_i 0 b_i} &= \frac{\Lambda}{(D + 2)} \delta_{a_i b_i} \end{aligned} \quad (4.28)$$

where $D = \sum_i d_i = \sum 2n_i$ is the total dimension of the base space, Λ is the (universal) cosmological constant of the CP^{n_i} , and the indices a_i label the coordinates on CP^{n_i} . (We are using the expression (4.22) for the Riemann tensor of CP^n , appropriately rescaled so that the cosmological constant is Λ .)

From (4.28) it follows that the non-zero components of the Weyl tensor are

$$\begin{aligned} \hat{C}_{a_i b_i c_i d_i} &= \Lambda \left[\frac{1}{d_i + 2} - \frac{1}{D + 2} \right] (\delta_{a_i c_i} \delta_{b_i d_i} - \delta_{a_i d_i} \delta_{b_i c_i} + \\ &\quad J_{a_i c_i} J_{b_i d_i} - J_{a_i d_i} J_{b_i c_i} + 2 J_{a_i b_i} J_{c_i d_i}), \\ \hat{C}_{a_i b_i a_j b_j} &= -\frac{2\Lambda}{D + 2} J_{a_i b_i} J_{a_j b_j}, \\ \hat{C}_{a_i a_j b_i b_j} &= -\frac{\Lambda}{D + 2} (\delta_{a_i b_i} \delta_{a_j b_j} + J_{a_i b_i} J_{a_j b_j}). \end{aligned} \quad (4.29)$$

The integrability conditions (4.26) for the existence of Killing spinors therefore become

$$\Gamma_{a_i b_i} \eta + J_{a_i c_i} J_{b_j d_j} \Gamma_{c_i d_j} \eta = 0, \quad (4.30)$$

$$\begin{aligned} (D - d_i) (\Gamma_{a_i b_i} + J_{a_i c_i} J_{b_i d_i} \Gamma_{c_i d_i} + J_{a_i b_i} J_{c_i d_i} \Gamma_{c_i d_i}) \eta \\ - (d_i + 2) J_{a_i b_i} \sum_{j \neq i} J_{c_j d_j} \Gamma_{c_j d_j} \eta = 0. \end{aligned} \quad (4.31)$$

One can show that (4.31) is implied by (4.30), and in fact the full set of independent conditions can be summarised succinctly as follows. Without loss of generality we can choose a basis for the CP^{n_i} spaces in which the orthonormal components of the Kähler forms are:

$$J_{12} = J_{34} = J_{56} = \cdots = +1, \quad (4.32)$$

with all other components being either zero, or implied by antisymmetry from the given ones. The conditions (4.30) and (4.31) can then be shown to be precisely equivalent to the conditions

$$\Gamma_{12} \eta = \Gamma_{34} \eta = \Gamma_{56} \eta = \cdots \Gamma_{D-1,D} \eta. \quad (4.33)$$

Since D is even, and the total bundle space has dimension $D + 1$, it follows that the spinors have $2^{D/2}$ components. There are $\frac{1}{2}D - 1$ equations in (4.33), each of which implies a halving of the original number of components, and so the final conclusion is that there are always 2 Killing spinors in these bundle spaces (real or complex, according to whether the spinors are Majorana or not). Special cases of this result that have appeared previously in the literature include the $U(1)$ bundles over $S^2 \times S^2$, $S^2 \times S^2 \times S^2$ and $CP^2 \times S^2$. We shall in general refer to all the $q_i = k_i/\ell$ Einstein spaces as ‘‘supersymmetric’’ spaces, although of course their Killing spinors are really only associated with supersymmetric compactifications in certain low-dimensional examples.

5 Consistency condition for Kaluza-Klein reductions

We saw in section 2 that in the cases of interest in supergravity reductions, a criterion for the consistency of the Kaluza-Klein reduction, and truncation to the massless gauge-boson sector, is that the Killing vectors \hat{K}^I associated with any gauge bosons that are to be retained must satisfy the condition that

$$Y(\hat{K}^I, \hat{K}^J) = \hat{K}^{Im} \hat{K}_m^J + \frac{1}{2m^2} (\hat{\nabla}^m \hat{K}^{In})(\hat{\nabla}_m \hat{K}_n^J) \equiv Y^{IJ} \quad (5.1)$$

should be constant, independent of the coordinates y of the internal space. Here, m is the related to the cosmological constant $\hat{\Lambda}$ of the internal Einstein space by $\hat{\Lambda} = D m^2$, where the dimension of the internal space is $D + 1$.

We begin by noting that the second term in (5.1) can be re-expressed more simply by using the following identity:

$$\begin{aligned} \hat{\square}(\hat{K}^m \hat{L}_m) &= \hat{K}^m \hat{\square} \hat{L}_m + \hat{L}^m \hat{\square} \hat{K}_m + 2(\hat{\nabla}^m \hat{K}^n)(\hat{\nabla}_m \hat{L}_n) \\ &= -2\hat{\Lambda} \hat{K}^m \hat{L}_m + 2(\hat{\nabla}^m \hat{K}^n)(\hat{\nabla}_m \hat{L}_n), \end{aligned} \quad (5.2)$$

for any pair of Killing vectors \hat{K}^m and \hat{L}^m , where we have made use of the fact that Killing vectors on an Einstein space with cosmological constant $\hat{\Lambda}$ satisfy the equation $\hat{\square}\hat{K}^m + \hat{\Lambda}\hat{K}^m = 0$. This allows us to express the second term in (5.1) in terms of $\hat{K}^m \hat{L}_m$:

$$(\hat{\nabla}^m \hat{K}^n)(\hat{\nabla}_m \hat{L}_n) = \frac{1}{2}\hat{\square}(\hat{K}^m \hat{L}_m) + \hat{\Lambda}\hat{K}^m \hat{L}_m. \quad (5.3)$$

Note that we just need the Laplacian \square on the base space here, since it is equal to the Laplacian $\hat{\square}$ in the bundle space when acting on scalars that are independent of the fibre coordinate z . The quantity $Y(\hat{K}, \hat{L})$ defined in (5.1), whose constancy is needed for consistency, is therefore expressible as

$$Y(\hat{K}, \hat{L}) = \frac{1}{2}(D+2)\hat{K}^m \hat{L}_m + \frac{D}{4\hat{\Lambda}}\hat{\square}(\hat{K}^m \hat{L}_m). \quad (5.4)$$

We shall refer to the criterion that $Y(\hat{K}, \hat{L})$ in (5.1) be constant as “The Consistency Condition” for short.

With our results from the previous sections we are now able to test the consistency condition in general, for any Einstein space \hat{M} that is constructed as a $U(1)$ bundle over a product of complex projective base spaces. Before doing so, we shall show that for *any* sphere S^n , with its standard round metric, all the $SO(n+1)$ Killing vectors satisfy the consistency condition. This is an important point not only for the discussion of Kaluza-Klein reductions on spheres themselves, but also we shall need to make use of this fact later in the section, when we examine the consistency condition in more general cases.

One way to prove that the full set of $SO(n+1)$ Killing vectors on the sphere S^n satisfy the consistency condition is by using the fact that there are always Killing spinors on the sphere, equal in number to the dimension of the spinors, that satisfy

$$\hat{\nabla}_m \eta^A - \frac{i}{2}m\Gamma_m \eta^A = 0. \quad (5.5)$$

From any pair of these, one can construct vectors $\hat{K}_m^{AB} = \bar{\eta}^A \Gamma_m \eta^B$, which can easily be seen to satisfy the Killing vector equation. One can also show that all the Killing vectors of $SO(n+1)$ are obtained by this means. Furthermore, it follows from (5.5) that $\hat{\nabla}_m \hat{K}_n^{AB} = i m \bar{\eta}^A \Gamma_{mn} \eta^B$. It is now relatively straightforward to show, using Fierz rearrangements, that the Killing vectors do indeed satisfy the consistency condition.

There is another way of showing that the full set of Killing vectors on the sphere satisfy the consistency condition, which is, perhaps, a little more geometrically appealing. We can describe the unit sphere S^n as the surface $x^A x^A = 1$ in \mathbb{R}^{n+1} , where x^A are Cartesian coordinates in \mathbb{R}^{n+1} . The Killing vectors on S^n are then given by

$$K_{AB} = x^A \partial_B - x^B \partial_A. \quad (5.6)$$

If we write $x^A = r u^A$, where the u^A satisfy $u^A u^A = 1$ and are coordinates on the unit S^n , and $r^2 = x^A x^A$, then the metric on \mathbb{R}^{n+1} is given by

$$ds^2(\mathbb{R}^{n+1}) = dr^2 + r^2 du^A du^A, \quad (5.7)$$

where $du^A du^A$ is the metric on the unit S^n . If we denote by g_{AB} the metric on the unit S^n , it is clear that it is related to the flat metric δ_{AB} on \mathbb{R}^{n+1} by

$$g_{AB} = \frac{1}{r^2} \left(\delta_{AB} - \frac{x^A x^B}{r^2} \right), \quad (5.8)$$

since this gives $g_{AB} dx^A dx^B = du^A du^A$. An elementary calculation then shows that the inner product between Killing vectors K_{AB} and K_{CD} given in (5.6), with respect to the metric g_{AB} , is

$$(K_{AB} \cdot K_{CD}) = \delta_{AC} u_B u_D + \delta_{BD} u_A u_C - \delta_{AD} u_B u_C - \delta_{BC} u_A u_D. \quad (5.9)$$

Now, the Laplacian on \mathbb{R}^{n+1} is related to the Laplacian on the unit S^n by

$$\square_{R^{n+1}} = \frac{1}{r^n} \frac{\partial}{\partial r} \left(r^n \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \square_{S^n}. \quad (5.10)$$

From (5.9), and $x^A = r u^A$, we shall have

$$\square_{R^{n+1}} \left(r^2 (K_{AB} \cdot K_{CD}) \right) = 4(\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}), \quad (5.11)$$

and hence using (5.10) we obtain

$$\square_{S^n} (K_{AB} \cdot K_{CD}) + 2(n+1) (K_{AB} \cdot K_{CD}) = 4(\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}). \quad (5.12)$$

Since the unit S^n has cosmological constant $(n+1)$, which corresponds to $m^2 = 1$ in (5.1), we finally arrive at the result that on the unit S^n

$$Y(K_{AB}, K_{BC}) = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}. \quad (5.13)$$

This shows that indeed all the $SO(n+1)$ Killing vectors on the sphere S^n satisfy the consistency condition.

We now turn to the case where the internal manifold is a general Einstein space that can be constructed as a $U(1)$ bundle over a product of complex projective spaces, of the kind we have discussed in the previous sections. In section 3, we derived the expression (3.30) for a Killing vector on the bundle space, and (3.32) for its expression as a 1-form. It is now straightforward to calculate the inner product between any two Killing vectors, which we

shall need for testing the consistency condition. Let us first establish the notation that we shall write the $U(1)$ Killing vector that generates translations along the fibres as

$$U \equiv \frac{\partial}{\partial z}. \quad (5.14)$$

It is easily seen that written as a 1-form, this is

$$U = c^2 (dz - A). \quad (5.15)$$

We shall use \hat{K}_i to denote a Killing vector lifted from the factor M_i in the base manifold. There are four different sectors to consider in the consistency condition, namely $Y(U, U)$, $Y(U, \hat{K}_i)$, $Y(\hat{K}_i, \hat{K}_j)$ (with $i \neq j$) and $Y(\hat{K}_i, \hat{L}_i)$ (where \hat{K}_i and \hat{L}_i are two Killing vectors in the *same* factor M_i in the base manifold).

Taking $Y(U, U)$ first we see from (5.14) and (5.15) that $U^m U_m = c^2 = \text{constant}$, and hence from (5.4) we shall have $Y(U, U) = \text{constant}$. So the $U(1)$ Killing vector by itself always satisfies the consistency condition.

Next, consider $Y(U, \hat{K}_i)$. From (5.14) and (3.32) we have

$$U^m \hat{K}_m^{(i)} = \alpha_i c^2 \psi^{(i)}, \quad (5.16)$$

and so from (5.4) we obtain

$$Y(U, \hat{K}_i) = \frac{\alpha_i c^2}{2\hat{\Lambda}} \left[(D + 2) \hat{\Lambda} - D \Lambda_i \right] \psi^{(i)}. \quad (5.17)$$

Since $\psi^{(i)}$ is never constant (it satisfies $\square \psi^{(i)} = -2\Lambda_i \psi^{(i)}$), it follows that for a Killing vector $\hat{K}^{(i)}$ coming from the base to be included in a consistent truncation as well as the $U(1)$ Killing vector U , the quantity in square brackets would have to vanish, *i.e.*

$$C_i \equiv (D + 2) \hat{\Lambda} - D \Lambda_i = 0. \quad (5.18)$$

We shall not analyse this condition extensively at this stage, since as we shall see later, more severe inconsistency problems generally occur in other sectors. We just note, however, that in view of the relation (3.40), consistency in this sector would require

$$\sum_j d_j \Lambda_j - D \Lambda_i = 0. \quad (5.19)$$

In particular, this would be satisfied if all the Λ_j were equal, $\Lambda_j = \Lambda$, since $\sum_j d_j = D$ (this is the case for all the spaces with $q_i = k_i/\ell$, *i.e.* the ones that admit 2 Killing spinors). However, we shall see below that the Killing vector \hat{K}_i will still run into other consistency problems in this case.

Moving on to the $Y(\hat{K}_i, \hat{K}_j)$ sector, where \hat{K}_i and \hat{K}_j come from different factors M_i and M_j in the base space, we find from (3.30) and (3.32) that the inner product for two such Killing vectors is

$$\hat{K}_i^m \hat{K}_{mj} = \alpha_i \alpha_j c^2 \psi^{(i)} \psi^{(j)}. \quad (5.20)$$

Since the two functions $\psi^{(i)}$ and $\psi^{(j)}$ are assumed to live in two different factors in the base space here, it follows that $\partial^a \psi^{(i)} \partial_a \psi^{(j)} = 0$, and hence, substituting into (5.4), we find

$$Y(\hat{K}_i, \hat{K}_j) = \frac{\alpha_i \alpha_j c^2}{2\hat{\Lambda}} \left[(D+2)\hat{\Lambda} - D(\Lambda_i + \Lambda_j) \right] \psi^{(i)} \psi^{(j)}. \quad (5.21)$$

Again, since the $\psi^{(i)}$ and $\psi^{(j)}$ functions are always non-constant, the only way for $Y(\hat{K}_i, \hat{K}_j)$ to be constant would be if the quantity in square brackets vanished, namely

$$C_{ij} \equiv (D+2)\hat{\Lambda} - D(\Lambda_i + \Lambda_j) = 0. \quad (5.22)$$

Again, without fully analysing this condition here we may note that in the cases of principal interest with $\Lambda_k = \Lambda$ for all k (the ‘‘supersymmetric’’ cases where there are 2 Killing spinors), equation (3.40) now allows us to deduce that

$$C_{ij} = -D\Lambda, \quad (5.23)$$

and so the consistency condition is not satisfied. Thus we already see that we could not include Killing vectors from both of two factors M_i and M_j in the base space, at least in the supersymmetric cases where all the Λ_k are equal.

The fourth sector to consider is when two Killing vectors \hat{K} and \hat{L} come from the same factor M_i in the base space. In order to avoid an unnecessary profusion of indices, we shall now suppress the ‘‘ i ’’ index that labels the particular factor in the product base manifold where the two Killing vectors are living. Thus the quantities \hat{K} , \hat{L} , ψ , $\tilde{\psi}$, d , α , Λ in the following discussion all refer to this specific factor in the base space.

Now, the calculation of the inner product of the gives the result

$$\hat{K}^m \hat{L}_m = \alpha^2 c^2 \psi \tilde{\psi} + \partial^a \psi \partial_a \tilde{\psi}, \quad (5.24)$$

where $K^a = J^{ab} \partial_b \psi$ and $L^a = J^{ab} \partial_b \tilde{\psi}$. Substituting into Y defined in (5.4), we now find

$$\begin{aligned} Y(\hat{K}, \hat{L}) = & \\ & \frac{1}{2\hat{\Lambda}} \left\{ \alpha^2 c^2 [(D+2)\hat{\Lambda} - 2D\Lambda] \psi \tilde{\psi} + [(D+2)\hat{\Lambda} + D(\alpha^2 c^2 - \Lambda)] \partial^a \psi \partial_a \tilde{\psi} \right. \\ & \left. + D(\nabla^a \nabla^b \psi)(\nabla_a \nabla_b \tilde{\psi}) \right\}. \end{aligned} \quad (5.25)$$

This equation can be simplified considerably, as follows. We may invoke the fact that if we consider the case where the base manifold has just a single factor $M_i = CP^{n_i}$, then the corresponding bundle space, with its Einstein metric, is the standard round metric on the sphere S^{2n_i+1} . Furthermore, we know that in this case all the Killing vectors on S^{2n_i+1} satisfy the consistency condition, as we discussed earlier. This, therefore, allows us to deduce that the scalars ψ and $\tilde{\psi}$ must satisfy equations such that (5.25) is constant when we take just the single factor M_i in the base space. In this case we shall have $D = d$ (the dimension of the single space M_i). Substituting into (5.25), we then learn that

$$-\frac{4\Lambda^2}{d+2}\psi\tilde{\psi} + \frac{4\Lambda}{d+2}\partial^a\psi\partial_a\tilde{\psi} + (\nabla^a\nabla^b\psi)(\nabla_a\nabla_b\tilde{\psi}) \quad (5.26)$$

must be a constant, for any choice of ψ and $\tilde{\psi}$ on M_i . This result⁸ for the eigenfunctions ψ on CP^n that they satisfy the condition that (5.26) is constant for any pair of such eigenfunctions. can now be fed back into (5.25) in the cases that really interest us, namely when there is more than one factor in the product base manifold. Specifically, we can use (5.26) in order to eliminate the $(\nabla^a\nabla^b\psi)(\nabla_a\nabla_b\tilde{\psi})$ terms in (5.25). Thus, we can deduce that consistency in this sector will be achieved only if

$$Q \equiv \partial^a\psi\partial_a\tilde{\psi} - \beta\psi\tilde{\psi} \quad (5.28)$$

is constant, where the constant β is given by

$$\beta = \frac{\frac{4D\Lambda^2}{d+2} - \alpha^2 c^2 [2D\Lambda - (D+2)\hat{\Lambda}]}{\frac{4D\Lambda}{d+2} - (D+2)\hat{\Lambda} - D(\alpha^2 c^2 - \Lambda)}. \quad (5.29)$$

Using (3.36) and (3.37), this can be rewritten as

$$\beta = \frac{4D\Lambda^2 - 2(\Lambda - \hat{\Lambda})[2D\Lambda - (D+2)\hat{\Lambda}](d+2)}{4D\Lambda - (d+2)[(D+2)\hat{\Lambda} + D(\Lambda - 2\hat{\Lambda})]}. \quad (5.30)$$

It is easiest to analyse this condition in the case where the Killing vector \hat{L} is taken to be the same as \hat{K} , since if we can show that $Y(\hat{K}, \hat{K})$ is not a constant, then that will show that no Killing vector from the base space can be retained in a consistent truncation. Let

⁸One can also prove this result directly, as follows. We know that any Killing vector K^a satisfies $\nabla_a\nabla_b K_c = R^d_{abc}K_d$. Since we have $K^a = J^{ab}\partial_b\psi$ here, and furthermore the Riemann tensor on CP^n is given by (4.22), we can conclude, after rescaling to cosmological constant Λ on CP^n , that

$$\nabla_a\nabla_b\nabla_c\psi = \frac{\Lambda}{d+2} \left[J_{ab}J_{cd}\partial^d\psi + J_{ac}J_{bd}\partial^d\psi - g_{ab}\partial_c\psi - g_{ac}\partial_b\psi - 2g_{bc}\partial_a\psi \right], \quad (5.27)$$

where $d = 2n$. After some simple further manipulations, the constancy of (5.26) follows.

us therefore just consider one scalar eigenfunction ψ , with $\tilde{\psi} = \psi$. Thus we wish to study whether the quantity

$$Q \equiv \partial^a \psi \partial_a \psi - \beta \psi^2 \quad (5.31)$$

can be constant. If Q is constant then $\nabla_a Q$ will be zero, and so we can follow the familiar strategy of integrating $(\nabla_a Q)^2$ over the factor $M_i = CP^{n_i}$ in the product base manifold, where the scalar eigenfunction ψ resides. If we can show that this integral is positive, then it will establish that Q is not constant, and hence that the gauge boson associated to the corresponding Killing vector cannot be retained in a consistent Kaluza-Klein reduction.

Using integrations by parts, and the equation $\square \psi = -2\Lambda \psi$, repeatedly, we can establish the following results:

$$\begin{aligned} \int \psi^2 |\nabla \psi|^2 &= \frac{2}{3} \Lambda \int \psi^4, \\ \int \nabla_a |\nabla \psi|^2 \nabla^a (\psi^2) &= \frac{8}{3} \Lambda^2 \int \psi^4 - 2 \int |\nabla \psi|^4, \\ \int \nabla_a |\nabla \psi|^2 \nabla^a |\nabla \psi|^2 &= \frac{8}{3} \Lambda^3 \int \psi^4 - \frac{2(d-2)}{d+2} \Lambda \int |\nabla \psi|^4, \end{aligned} \quad (5.32)$$

where $|\nabla \psi|^2 \equiv \nabla_a \psi \nabla^a \psi$ and $|\nabla \psi|^4 \equiv (|\nabla \psi|^2)^2$. (We have used the relation (5.27) in obtaining the last of these three equations.) Using these results, we find that

$$\int |\nabla Q|^2 = \frac{8}{3} \Lambda (\beta - \Lambda)^2 \int \psi^4 + 2 \left(2\beta - \frac{d-2}{d+2} \Lambda \right) \int |\nabla \psi|^4. \quad (5.33)$$

Using this, it is possible to show that, except for “trivial” cases that we shall discuss below, the quantity Q can never be constant for any of the eigenfunctions ψ associated with the Killing vectors of the $SU(n_i + 1)$ factors in the isometry group of the bundle space. We shall first discuss the “supersymmetric” cases, where the winding numbers q_i satisfy $q_i = k_i/\ell$, since the proof is very simple in these cases, and furthermore they are the examples of principal physical interest. After that, we shall present a complete analysis for all possible choices of winding numbers.

As we saw in section 3, when $q_i = k_i/\ell$ the cosmological constants of all the CP^{n_i} factors in the base space are equal, as are the constants α_i ; they are given by (3.44) and (3.45). Substituting these into (5.29) we find $\beta = \Lambda$, and so (5.33) gives

$$\int |\nabla Q|^2 = \frac{2\Lambda(d+6)}{(d+2)} \int |\nabla \psi|^4. \quad (5.34)$$

The right-hand side is manifestly positive, and so the result that Q cannot be constant follows.

For the general (non-supersymmetric) case with arbitrary winding numbers q_i , consider first the situation when the factor M_i in the base space where ψ resides is CP^1 . In this

particular case, because CP^1 is the sphere S^2 , it follows that the three eigenfunctions ψ that generate the $SO(3)$ Killing vectors actually satisfy the equation $\nabla_a \nabla_b \psi = -\Lambda g_{ab} \psi$, and from this it follows that on CP^1 we have

$$\int |\nabla \psi|^4 = \frac{8}{3} \Lambda^2 \int \psi^4. \quad (5.35)$$

Substituting this into (5.33) gives

$$\int |\nabla Q|^2 = \frac{8}{3} \Lambda (\beta + \Lambda)^2 \int \psi^4. \quad (5.36)$$

Thus we see that in this case it must be that Q is constant if and only if $\beta = -\Lambda$. It is easy to see from the equations (3.35), (3.36), (3.37) and (5.29) that this can happen only in the extreme case where the fibres in the $U(1)$ bundle have a non-zero winding number *only* over the S^2 factor in the base space where ψ resides. But in this extreme case the total space is simply the direct product of S^3 times the remaining CP^{n_i} factors in the base. Not surprisingly, since S^3 is a group manifold, it has Killing vectors for which the associated quantity Q will be constant. (Since any given Killing vector is associated with a left-translation or right-translation under $SU(2)$.) Aside from this extreme case, which is certainly not the one of interest to us in this paper, we see that Q can never be constant.

Next, consider the case where the eigenfunction ψ lives in a CP^2 factor in the base space. It is necessary, again, to determine the relation between $\int |\nabla \psi|^4$ and $\int \psi^4$. Clearly this will be of the form

$$\int |\nabla \psi|^4 = c \Lambda^2 \int \psi^4, \quad (5.37)$$

where c is a pure (dimensionless) number. It is evident from the expressions (4.18) or (4.19) for ψ that the two integrals on CP^2 must be expressible in terms of $SU(3)$ -invariant quartic polynomials built from the traceless Hermitean tensor T_A^B . Since there is no independent fourth-order Casimir for $SU(3)$, it must be that both integrals in (5.37) for CP^2 are pure numbers times $(T_A^B T_B^A)^2$, the numbers being independent of the choice of T_A^B . Thus the constant c can be determined by evaluating the two sides of (5.37) for any convenient choice of eigenfunction ψ . From (4.19), a simple choice is to take the ψ corresponding to $T_\alpha^\beta = \delta_\alpha^\beta$, which implies $T_0^0 = -2$, with all other components of T_A^B zero. This gives

$$\psi = 1 - 3 f^{-1}, \quad (5.38)$$

where f is given in (4.2). It is easy to substitute this into (5.37), leading to the result that

$$c = 2. \quad (5.39)$$

Finally, using this result in (5.33), with $n = \frac{1}{2}d$, we arrive at the following:

$$\int |\nabla Q|^2 = \frac{8}{3} \Lambda \left[(\beta + \frac{1}{2}\Lambda)^2 + \frac{1}{4}\Lambda^2 \right] \int \psi^4, \quad (5.40)$$

which shows that Q can never be constant in this case.

Finally, we can consider the general case where ψ lives in a CP^n factor in the base space. Now the calculation is a little more involved, since the ratio of $\int |\nabla \psi|^4$ to $\int \psi^4$ depends on the specific choice of eigenfunction ψ , when $n \geq 3$. In order to achieve the best chance of having $\int |\nabla Q|^2$ be zero, one wants the ratio of $\int |\nabla \psi|^4$ to $\int \psi^4$ to be as large as possible, since then the (possibly negative) second term on the right-hand side of (5.33) has the best chance to outweigh the always-positive contribution from the first term on the right-hand side. In the Appendix we present some calculations that provide a determination of the largest value of this ratio; see (B.17) and (B.18). Thus from (5.33) we find that when n is odd, we shall have

$$\int |\nabla Q|^2 \geq \frac{8}{3} \Lambda \left(\beta + \frac{2\Lambda}{n+1} \right)^2 \int \psi^4, \quad (5.41)$$

whilst when n is even we instead find

$$\int |\nabla Q|^2 \geq \frac{8}{3} \Lambda \left[\left(\beta + \frac{2n\Lambda}{n^2+n+2} \right)^2 + \frac{4(n+2)}{(n^2+n+2)^2} \right] \int \psi^4, \quad (5.42)$$

From these results we see that Q can never be constant when n is even. When n is odd instead, we see that Q can be constant if and only if

$$\beta = -\frac{2\Lambda}{n+1}. \quad (5.43)$$

Now from (3.36) and (3.37) it immediately follows that if we define $x \equiv \hat{\Lambda}/\Lambda$, then

$$\frac{d}{d+2} \leq x \leq 1. \quad (5.44)$$

The lower limit is saturated if the $U(1)$ fibres wind only over the chosen CP^n factor in the base space, whilst the upper limit is saturated if instead the $U(1)$ fibres have zero winding number over the chosen CP^n factor. Using (5.30), we find that

$$\beta + \frac{4\Lambda}{d+2} = \Lambda \frac{-2(D+2)(d+2)^2 x^2 + 2(d+2)(3Dd+8D+2d)x - 4dD(d+4)}{(d+2)[(D-2)(d+2)x - D(d-2)]}. \quad (5.45)$$

The denominator is positive for all x in the interval (5.44), and the numerator has no extremum in this interval. It then follows that we shall have

$$\beta \geq -\frac{4\Lambda}{d+2}, \quad (5.46)$$

with equality being achieved only if $x = d/(d+2)$. Since $d = 2n$, we conclude from this and (5.43) that Q can be constant only in the extreme case where the $U(1)$ fibres wind purely over the CP^n factor in the base space in which the eigenfunction ψ resides.

The reason for the occurrence of these exceptional cases where Q can be constant is the following. When n is odd, say $n = 2q + 1$, and the fibres wind only over the CP^{2q+1} factor in the base manifold, the total space is the direct product of S^{4q+3} with the other CP^{n_i} factors in the base space. Now the sphere S^{4q+3} can be described as an $SU(2)$ bundle over the quaternionic projective space HP^q . Consequently, an $SU(2)$ subgroup of the $SO(4q+4)$ isometry group of the sphere corresponds to left translations by $SU(2)$ on the $SU(2)$ fibres, and therefore the associated $SU(2)$ Killing vectors K^I will necessarily have the property that $K^I \cdot K^J = \text{constant}$, and so they will be associated with eigenfunctions ψ on CP^{2q+1} that satisfy the condition $Q = \text{constant}$. It is these Killing vectors that are being “detected” by the saturation of the bound (5.41).

These exceptional cases are higher-dimensional generalisations of the exception arising for $n = 1$, with the fibres winding only over the CP^1 factor to give S^3 , which we discussed previously. Again they are “trivial,” from the point of view of our analysis of compactifications, since we are not particularly interested in cases where the internal space is a direct product of a sphere S^{4q+3} with a Kähler space. Nonetheless, it is reassuring to find that our rather intricate general analysis has indeed, as it should, detected these slightly obscure exceptions to the general rule.

With these results, we have proved that the non-abelian Killing vectors on the $U(1)$ bundle spaces over any product of CP^{n_i} factors in the base space will never satisfy the consistency requirement that Y^{IJ} in (5.1) is a constant.⁹ This means that the associated Kaluza-Klein Yang-Mills fields associated with the non-abelian part of the symmetry group cannot be consistently retained in a massless truncation. In particular, this proves that of the $U(1) \times SU(2) \times SU(2)$ Yang-Mills fields in the $Q(1,1)$ compactification of the type IIB theory to $D = 5$, the $SU(2) \times SU(2)$ fields cannot be retained in a consistent massless truncation.

⁹Except in the previously-discussed trivial cases of $SU(2)$ Killing vectors in the S^{4q+3} factors in a bundle space where the fibres wind only over a CP^{2q+1} base-space factor.

6 Conclusions

In this paper, we have studied a necessary condition for the occurrence of a consistent Kaluza-Klein reduction on an internal Einstein manifold, in which all the Yang-Mills fields associated with the isometry group of the compactifying space are retained in a massless truncation. This condition, that the quantity Y^{IJ} defined in (5.1) should be constant, is of rather general relevance in all the known non-trivial consistent Kaluza-Klein reductions. In particular, this consistency criterion is satisfied by all the Killing vectors on a sphere, of arbitrary dimension. Our principal goal in this paper has been to show that the consistency criterion is never satisfied by the non-abelian $SU(n_i + 1)$ Killing vectors in the isometry groups of the spaces $Q_{n_1 \dots n_N}^{q_1 \dots q_N}$, which are defined as $U(1)$ bundles over the product $\prod_i CP^{n_i}$ of complex-projective spaces CP^{n_i} , with winding numbers q_i . In particular, this shows that space Q_{11}^{11} (sometimes known as T^{11}), the $U(1)$ bundle over $S^2 \times S^2$, does not allow a consistent Kaluza-Klein reduction of type IIB supergravity in which the non-abelian Yang-Mills fields of its $SU(2) \times SU(2) \times U(1)$ isometry group are retained in a massless truncation. Likewise, the compactifications of $D = 11$ supergravity on the $U(1)$ bundles over $S^2 \times S^2 \times S^2$ and over $CP^2 \times S^2$ do not allow the retention of the corresponding non-abelian Yang-Mills fields in massless truncations. These facts will be of relevance in the AdS/CFT correspondence [34, 35, 36], where it should turn out that certain correlation functions involving products of single massive operators with massless ones will correspondingly be non-zero in these cases (see, for example, [32]).

We have set our proof of the inconsistency of the full massless truncations in these cases in a more general context, in which we show in general that the non-abelian Killing vectors on the bundle spaces $Q_{n_1 \dots n_N}^{q_1 \dots q_N}$ do not satisfy the consistency criterion that all Killing vectors on all spheres satisfy. In order to show this, we have made a detailed analysis that should also be of more general utility. In particular, we studied the lifting of Killing vectors from an arbitrary base manifold to a $U(1)$ bundle over the base, and then we specialised to the case where the base is a Kähler-Einstein space, or a product of Kähler-Einstein spaces. In such cases, more complete results can be obtained, based on the fact that any Killing vector in the base can be expressed in terms of a certain eigenfunction of the scalar Laplacian.

We then turned to the cases of principal interest, where the base space is the product of complex projective spaces CP^{n_i} . We made a study of the Fubini-Study metrics, and in an appendix we obtained a rather useful iterative construction for real metrics on CP^n . We showed that all of the bundle spaces $Q_{n_1 \dots n_N}^{q_1 \dots q_N}$ can be given Einstein metrics, provided only that all the winding numbers q_i do not vanish. We also showed that in the special

case where $q_i = (n_i + 1)/\ell$, where ℓ is the greatest common divisor of the $(n_i + 1)$, the Einstein spaces all admit two Killing spinors. These cases, for Q_{11}^{11} , Q_{111}^{111} and Q_{21}^{32} , are the ones of principal interest in the context of supergravity compactifications, since they imply the existence of unbroken supersymmetries.

We showed also that the question of whether the non-abelian Killing vectors of the $U(1) \times \prod_i SU(n_i + 1)$ isometry group of $Q_{n_1 \dots n_N}^{q_1 \dots q_N}$ satisfy the consistency criterion in (5.1) can be reduced to the question of whether the scalar eigenfunctions on CP^n that are related to its Killing vectors satisfy certain integral bounds. We studied these bounds in detail, and used these to obtain our proofs of the inconsistency of the Kaluza-Klein reductions.

Acknowledgements

We are grateful to Mirjam Cvetič, Hong Lü, Arta Sadrzadeh, Kelly Stelle and Tuan Tran for helpful discussions. C.N.P. thanks the Caltech-USC Center for Theoretical Physics for hospitality during the completion of this work.

Appendices

A An iterative construction of CP^n

On occasion, it is helpful to have a real expression for the Fubini-Study metric on CP^n available. This is easily done for low-dimensional examples by making specific adapted coordinate choices (see, for example, [33] for an explicit real metric on CP^2). In general, we can give an elegant iterative construction for the metric on CP^n in terms of the metric on CP^{n-1} .

We take as our starting point the standard Fubini-Study metric (4.6) on CP^n , and write the inhomogeneous coordinates ζ^α as

$$\zeta^\alpha = \tan \xi u^\alpha, \quad \text{with} \quad u^\alpha \bar{u}^{\bar{\alpha}} = 1. \quad (\text{A.1})$$

With this coordinate redefinition the CP^n metric (4.6) becomes

$$d\Sigma_n^2 = d\xi^2 + \sin^2 \xi du^\alpha d\bar{u}^{\bar{\alpha}} - \sin^4 \xi |\bar{u}^{\bar{\alpha}} du^\alpha|^2. \quad (\text{A.2})$$

Noting that the n quantities u^α are themselves complex coordinates on C^n , subject to the constraint $u^\alpha \bar{u}^{\bar{\alpha}} = 1$, we can follow the same strategy as in the original CP^n construction,

by introducing $(n - 1)$ inhomogeneous coordinates v^i , with $1 \leq i \leq n - 1$, defined by

$$v^i = \frac{u^i}{u^n}, \quad (\text{A.3})$$

where u^n here denotes the n 'th of the coordinates u^α . In addition, we define

$$u^n = |u^n| e^{i\tilde{\tau}}. \quad (\text{A.4})$$

After a little calculation, we see that the metric (4.6) on CP^n now takes the form

$$d\Sigma_n^2 = d\xi^2 + \sin^2 \xi \cos^2 \xi (d\tilde{\tau} + \tilde{B})^2 + \sin^2 \xi d\Sigma_{n-1}^2, \quad (\text{A.5})$$

where $d\Sigma_{n-1}^2$ is the Fubini-Study metric on the unit CP^{n-1} , and \tilde{B} is a potential for the Kähler form of CP^{n-1} :

$$\begin{aligned} d\Sigma_{n-1}^2 &= \tilde{f}^{-1} dv^i d\bar{v}^i - \tilde{f}^{-2} |\bar{v}^i dv^i|^2, \quad \tilde{f} = 1 + v^i \bar{v}^i, \\ \tilde{B} &= \frac{1}{2} i \tilde{f}^{-1} (v^i d\bar{v}^i - \bar{v}^i dv^i). \end{aligned} \quad (\text{A.6})$$

Thus (A.5) gives us an iterative construction of the Fubini-Study metric on the unit CP^n in terms of the Fubini-Study metric on the unit CP^{n-1} . (In fact the metric in CP^2 obtained in [33] is precisely of this form, with the metric on CP^1 being the standard metric on the 2-sphere.) Note that the potential B for CP^n , appearing in (4.8), is given in terms of the analogous potential \tilde{B} for CP^{n-1} by

$$B = \sin^2 \xi (d\tilde{\tau} + \tilde{B}). \quad (\text{A.7})$$

The function f appearing in (4.2) is given by

$$f = \sec^2 \xi. \quad (\text{A.8})$$

B Inequalities on CP^n

In section 5, we show that the gauge boson associated with any Killing vector on a factor CP^n in the base manifold whose associated scalar harmonic ψ has a Q , as defined in (5.31), that is non-constant, cannot be retained in a consistent massless Kaluza-Klein reduction. In this appendix we derive some inequalities involving the integrals $\int \psi^4$ and $\int |\nabla \psi|^4$ appearing in (5.33), which are used in the calculations in section 5.

In terms of the construction (4.18) or (4.19) for the eigenfunctions ψ , it is clear that the integrals $\int \psi^4$ and $\int |\nabla \psi|^4$ must necessarily give rise to quartic $SU(n + 1)$ invariants built from the traceless Hermitean tensor $T_A{}^B$. If we define

$$I_2 \equiv T_A{}^B T_B{}^A, \quad I_4 \equiv T_A{}^B T_B{}^C T_C{}^D T_D{}^A, \quad (\text{B.1})$$

it follows therefore that on CP^n we must have

$$\int \psi^4 = a (I_2)^2 + b I_4, \quad \int |\nabla \psi|^4 = \tilde{a} (I_2)^2 + \tilde{b} I_4, \quad (\text{B.2})$$

for pure numbers a , b , \tilde{a} and \tilde{b} that are dependent only on the value of n . In order to determine these constants, it suffices to consider just two special cases of eigenfunctions ψ that have different values for the ratio $I_4/(I_2)^2$.

A convenient choice for the two eigenfunctions ψ_1 and ψ_2 is as follows. For ψ_1 , we take $T_\alpha^\beta = \delta_\alpha^\beta$, $T_0^0 = -n$, with all other components of T_{AB} vanishing. For ψ_2 , we take instead $T_n^0 = T_0^n = \frac{1}{2}$, with all other components vanishing. (Here “ n ” indicates that α takes the value $\alpha = n$.) For these two special cases the invariants I_2 and I_4 are given by:

$$\begin{aligned} \psi_1 : \quad I_2 &= n(n+1), & I_4 &= n(n^3+1), \\ \psi_2 : \quad I_2 &= \frac{1}{2}, & I_4 &= \frac{1}{8}. \end{aligned} \quad (\text{B.3})$$

Thus when $n \geq 2$, we see that $I_4/(I_2)^2$ is different for the two eigenfunctions, and so by evaluating the integrals in (B.2), we shall be able to determine a , b , \tilde{a} and \tilde{b} .

In order to evaluate the integrals, it is convenient to make use of the iterative construction of CP^n metrics that we obtained in Appendix A. Specifically, we iterate twice, to give

$$d\Sigma_n^2 = d\xi^2 + \sin^2 \xi \cos^2 \xi (d\tilde{\tau} + \tilde{B})^2 + \sin^2 \xi \left(d\lambda^2 + \sin^2 \lambda \cos^2 \lambda (dz + C)^2 + \sin^2 \lambda d\Sigma_{n-2}^2 \right). \quad (\text{B.4})$$

(Our notation should be self-evident, by comparing with the construction in Appendix A.) The two eigenfunctions ψ_1 and ψ_2 can then be seen to be given by

$$\psi_1 = 1 - (n+1) \cos^2 \xi, \quad \psi_2 = \sin \xi \cos \xi \cos \lambda \cos \tilde{\tau}. \quad (\text{B.5})$$

Other relevant points are that the determinant of the metric (B.4) is given by

$$\sqrt{g_n} = (\sin \xi)^{2n-1} \cos \xi (\sin \lambda)^{2n-3} \cos \lambda \sqrt{g_{n-2}}, \quad (\text{B.6})$$

where g_{n-2} is the determinant of the metric $d\Sigma_{n-2}^2$ on CP^{n-2} . Furthermore, the relevant components of the inverse metric are given by

$$g^{\xi\xi} = 1, \quad g^{\lambda\lambda} = \frac{1}{\sin^2 \xi}, \quad g^{\tilde{\tau}\tilde{\tau}} = \frac{\sec^2 \xi + \tan^2 \lambda}{\sin^2 \xi}. \quad (\text{B.7})$$

For functions ϕ of ξ , λ and $\tilde{\tau}$ only, we have

$$|\nabla \phi|^2 = \left(\frac{\partial \phi}{\partial \xi} \right)^2 + \frac{1}{\sin^2 \xi} \left(\frac{\partial \phi}{\partial \lambda} \right)^2 + \frac{\sec^2 \xi + \tan^2 \lambda}{\sin^2 \xi} \left(\frac{\partial \phi}{\partial \tilde{\tau}} \right)^2. \quad (\text{B.8})$$

As a final preliminary, we note from (4.4) that since the unit sphere S^{2n+1} has volume $\Omega_{2n+1} = 2\pi^{n+1}/\Gamma(n+1)$, it follows that the unit CP^n has volume Σ_n given by

$$\Sigma_n = \frac{\pi^n}{\Gamma(n+1)}. \quad (\text{B.9})$$

It is now straightforward to evaluate all the necessary integrals, and thus to determine the constants a , b , \tilde{a} and \tilde{b} appearing in (B.2). We find that for the unit CP^n , with $n \geq 2$, we shall have

$$\begin{aligned} \int \psi^4 &= \frac{3\pi^{n-1}}{2\Gamma(n+5)} \left[(I_2)^2 + 2I_4 \right], \\ \int |\nabla \psi|^4 &= \frac{8\pi^{n-1}}{\Gamma(n+5)} \left[(n^2 + 5n + 7)(I_2)^2 + (n+1)(n+2)I_4 \right]. \end{aligned} \quad (\text{B.10})$$

For the discussion in section 5, it turns out that we need to know the largest possible value that the ratio $(\int |\nabla \psi|^4)/(\int \psi^4)$ can attain. It is easy to see from (B.10) that this will occur for a tensor $T_A{}^B$ that gives the smallest possible value of $I_4/(I_2)^2$. To determine this value, let the (real) eigenvalues of $T_A{}^B$ be λ_A . Tracelessness implies that $\sum_A \lambda_A = 0$. If we solve for the eigenvalue λ_0 in terms of the λ_α for $1 \leq \alpha \leq n$, we shall therefore have

$$I_2 = \sum_\alpha \lambda_\alpha^2 + (\sum_\alpha \lambda_\alpha)^2, \quad I_4 = \sum_\alpha \lambda_\alpha^4 + (\sum_\alpha \lambda_\alpha)^4, \quad (\text{B.11})$$

and so the ratio $R \equiv I_4/(I_2)^2$ is extremised when the λ_α satisfy

$$\lambda_\alpha^3 I_2 - \lambda_\alpha I_4 + (\sum_\beta \lambda_\beta)^3 I_2 - \sum_\beta \lambda_\beta I_4 = 0. \quad (\text{B.12})$$

If we suppose that two of the extremising eigenvalues, say λ_α and λ_β , are unequal, then by subtracting their equations (B.12) we find that

$$\lambda_\alpha^2 + \lambda_\alpha \lambda_\beta + \lambda_\beta^2 = \frac{I_4}{I_2}. \quad (\text{B.13})$$

If a third eigenvalue, say λ_γ , is unequal to both λ_α and λ_β , then by subtractions we can see that

$$\lambda_\alpha + \lambda_\beta + \lambda_\gamma = 0. \quad (\text{B.14})$$

Finally, if we suppose that a fourth eigenvalue λ_δ is unequal to all of the previous three, then by subtractions we arrive at the contradiction that $\lambda_\delta = \lambda_\alpha$. Therefore any set of λ_α that extremises the ratio R can involve at most three different values.

It now becomes rather straightforward to find the extrema, and in particular, to identify the global minima. There are two distinct cases, depending upon whether n is even or odd.

We find that the minimum is achieved for

$$\begin{aligned} n = 2q + 1 : \quad \lambda_0 = \lambda_1 = \cdots = \lambda_q = \lambda, \quad \lambda_{q+1} = \lambda_{q+2} = \cdots = \lambda_{2q+1} = -\lambda, \quad (B.15) \\ n = 2q : \quad \lambda_0 = \lambda_1 = \cdots = \lambda_q = \lambda, \quad \lambda_{q+1} = \lambda_{q+2} = \cdots = \lambda_{2q} = -\frac{n+2}{n} \lambda. \end{aligned}$$

(Of course in each case there are symmetry-related minima corresponding to permuting the eigenvalues. The minimisation occurs when the set of eigenvalues λ_i divides into two subsets that are as nearly as possible equal in size, within each of which all eigenvalues are equal. This 50/50 partitioning is exact only if n is odd, since the total number of eigenvalues λ_i is then even.) Thus we find that the following inequalities hold:

$$\begin{aligned} n = 2q + 1 : \quad \frac{I_4}{I_2^2} &\geq \frac{1}{n+1}, \\ n = 2q : \quad \frac{I_4}{I_2^2} &\geq \frac{n^2 + 2n + 4}{n(n+1)(n+2)}. \end{aligned} \quad (B.16)$$

Substituting into (B.10), and reinstating the cosmological constant Λ by the appropriate constant rescaling, we then find that

$$\int |\nabla \psi|^4 \leq c_{\max} \int \psi^4, \quad (B.17)$$

with

$$\begin{aligned} n = 2q + 1 : \quad c_{\max} &= \frac{4(n+3)\Lambda^2}{3(n+1)}, \\ n = 2q : \quad c_{\max} &= \frac{4(n+1)(n+2)\Lambda^2}{3(n^2+n+2)}. \end{aligned} \quad (B.18)$$

These inequalities are used in section 5.

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